UNIVERSITY OF MARIBOR

FACULTY OF NATURAL SCIENCES AND MATHEMATICS

Department of Mathematics and Computer Sciences

MASTER'S THESIS

Petra Fic

Maribor, 2019

UNIVERSITY OF MARIBOR
FACULTY OF NATURAL SCIENCES AND MATHEMATICS
Department of Mathematics and Computer Sciences

Master's thesis

TECHNOLOGICAL VALLEY OF DEATH AS AN EMERGENT EVOLUTIONARY PHENOMENON

in the Master's study of program Mathematics

Advisor:
Drago Bokal

Candidate:
Petra Fic

Maribor, 2019

UNIVERZA V MARIBORU

FAKULTETA ZA NARAVOSLOVJE IN MATEMATIKO

Oddelek za matematiko in računalništvo

Magistrsko delo

TEHNOLOŠKA DOLINA SMRTI KOT SAMOPORARAJOČI EVOLUCIJSKI FENOMEN

na študijskem programu 2. stopnje Matematika

Mentor: Drago Bokal

Kandidatka: Petra Fic

Maribor, 2019

ACKNOWLEDGEMENTS

Mathematics is the queen of science.
Carl Friedrich Gauss

Firstly, I would like to express my sincere gratitude to my thesis advisor dr. Drago Bokal. The door to Prof. Bokal's office was always open whenever I ran into a trouble spot or had a question about my research or writing. Without his passionate participation and input, the research could not have been successfully conducted.

I would like to thank my parents, Valerija and Dragan Fic, to my sister, Antonija Fic, and to my grandparents, Katarina and Vladimir Horvatić, for providing me with unfailing support and continuous encouragement throughout my years of study and through the process of researching and writing this thesis. This accomplishment would not have been possible without them. Thank you.

Finally, I must express my very profound gratitude to my future husband, Davor Žagar, who has supported me throughout entire process, both by keeping me harmonious and helping me putting pieces together. I will be grateful forever for your love.

Thank You all very much.

UNIVERSITY OF MARIBOR

FACULTY OF NATURAL SCIENCES AND MATHEMATICS

Department of Mathematics and Computer Sciences

STATEMENT OF AUTHORSHIP

I, Petra Fic, born on September 23, 1993, student of the Faculty of Natural Sciences and Mathematics on University of Maribor, in the Master's study of program Mathematics, hereby declare that I am the sole author of the master's thesis

TECHNOLOGICAL VALLEY OF DEATH AS AN EMERGENT EVOLUTIONARY PHENOMENON

under the supervision of dr. Drago Bokal. The sources and bibliography in this master's thesis are correctly stated and the texts are not listed without the authors' reference.

Maribor, 10^{th} October, 2019.

Petra Fic

Tehnološka dolina smrti kot samoporarajoči evolucijski fenomen
program magistrskega dela

Prepričanje, da naravna selekcija podpira veristična zaznavanja, t. j. tista, ki bolj natančno prikazujejo okolje, je razširjeno tudi med raziskovalci. Utemeljeno je s predpostavko, da so organizmi, katerih zaznave so bolj veristične, tudi bolj uspešni v okolju [1]. Ta predpostavka je bila testirana z uporabo standardnih orodij evolucijske teorije iger v poenostavljenem okolju. Rezultat je bil, da bolj veristične percepcije niso nujno bolj uspešne. Veristične zaznave v pretežnem delu prostora parametrov izumrejo v tekmi s poenostavljenimi zaznavami, ki vodijo v adaptivno vedenje, prilagojeno naravni selekciji v danem okolju [2]. V magistrskem delu povzamemo rezultate omenjenih bazičnih raziskav, ki jih nadgradimo s primerjavo vseh možnih strategij, ki smo jih razdelili v dve skupini: tiste, katerih zaznave temeljijo na koristnosti virov in tiste, ki temeljijo na veristični zaznavi količine virov. Navedeni taksonomski pristop je podlaga za sistematično primerjavo vseh možnih kombinacij strategij v vseh možnih okoljskih kontekstih (število virov, stabilnost okolja in število teritorijev).

Osnovni viri:

1. D. D. Hoffman, The interface theory of perception, Object Categorization: Computer and Human Vision Perspective, Cambridge University Press, New York (2009) str. 148–165.
2. M. H. Kalos, P. A. Whitlock, *Monte Carlo Methods, Second, Revised and Enlarged Edition*, Wiley-VCH, Weinheim, 2008.
3. J. T. Mark, B. B. Marion, D. D. Hoffman, Natural selection and veridical perceptions, Journal of Theoretical Biology, 266 (2010), str. 504–515.
4. J. W. Weibull, *Evolutionary game theory*, MIT press, Massachusetts, 1997.

Drago Bokal

FIC, P.: Technological valley of death as an emergent evolutionary phenomenon.
Master's Thesis, University of Maribor, Faculty of Natural Sciences and Mathematics, Department of Mathematics and Computer Science, 2019.

ABSTRACT

Perceptual researchers often argue that natural selection supports veridical perceptions, respectively those that accurately reflect the environment. They also claim that beings whose perceptions are truer are also more fit. This assumption was tested using standard tools of evolutionary game theory in a simple environment. The result was that more veridical perceptions are not necessarily more successful. In the majority of the parameter space, veridical perceptions are extinct in competition with simplified perceptions, based on adaptive behavior in a given environment. In the thesis, we build upon mentioned territorial games introduced by Mark, Marion, and Hoffman in 2010, and extend four of their territory perception and selection strategies with two novel ones that together constitute a model of technological readiness valley of death. Whenever utility of a resource is not monotonous in the amount of that resource, the technological valley of death emerges. While the development of the science behind these models is in its infancy, modeling and understanding the phenomenon may shed light on progress and related phenomena in society.

Keywords: evolution, perception, utility, Monte Carlo simulation, game theory.

Math. Subj. Class. (2010): 91A22 Evolutionary games,
22F10 Measurable group actions,
92D25 Population dynamics,
60B05 Probability measures on topological spaces.

FIC, P.: Tehnološka dolina smrti kot samoporarajoči evolucijski fenomen. Magistrsko delo, Univerza v Mariboru, Fakulteta za naravoslovje in matematiko, Oddelek za matematiko in računalništvo, 2019.

IZVLEČEK

Raziskovalci zaznav pogosto menijo, da naravna selekcija podpira veristična zaznavanja, oziroma tista, ki bolj natančno prikazujejo okolje. Trdijo tudi, da so bitja, katerih dojemanja so bolj resnična, tudi bolj uspešna. Ta predpostavka je bila testirana z uporabo standardnih orodij evolucijske teorije iger za enostavno okolje. Rezultat je pokazal, da bolj verodostojne percepcije niso tudi bolj uspešne. Zaradi naravne selekcije lahko veristične strategije izumrejo v tekmi s strategijami, ki so prilagojene na zaznavanje neposredne koristnosti možnih odločitev. Magistrsko delo gradi na omenjenih teritorialnih igrah, ki so jih uvedli Mark, Marion in Hoffman v letu 2010. Njihove štiri percepcijske strategije selekcije teritorijev razširimo z dvema novima, ki skupaj tvorita model tehnološke doline smrti. Domnevamo, da se, kadar koristnost vira ni monotona funkcija količine tega vira, pojavi tehnološka dolina smrti. Medtem ko so raziskave tega fenomena še v povojih, lahko modeliranje in razumevanje tega pojava osvetli napredek in z njim povezane pojave v družbi.

Ključne besede: evolucija, zaznave, koristnost, Monte Carlo simulacije, teorija iger.

Math. Subj. Class. (2010): 91A22 Evolucijske igre,
22F10 Merljivi ukrepi grup,
92D25 Dinamika populacij,
60B05 Verjetnosne mere na na topoloških prostorih.

Contents

Introduction

In this master's thesis, we show how mathematical modelling can be applied to model biological phenomena, and then how abstract models of biological phenomena can be applied to reason about our understanding of life. We will ask about how useful is it that our perceptions about world are accurate, veridical, in the sense of our models being isomorphic to the factual structure of the world.

The relationship between our perception and the environment is explored by theory of perceptions (D. Heyer, R. Mausfeld, [12], G. Radnitzky, [32], R. Schwartz, [35]). Several results from 2010 onwards state that veridical perceptions of the world have little if any chance of surviving evolutionary competition, which favors utilitaristic interface-like perceptions, optimized to perceive the utility derived from the actions in a given environment (D. D. Hoffman, [13]). Subsequent proof of the Invention of Symmetry Theorem (D. D. Hoffman, M. Singh, C. Prakash, [14]) stated that an observer who perceives the symmetry of the world, may simply be deceived by the symmetry emanating from the compositum of perceptions of the utility of results of actions upon the world, which would exhibit the stated structure, but would be just an optimal interface to the true structure of the world that could have a completely different structure. Scientific community responded grimly to this pessimistic view (D. D. Hoffman, M. Singh, C. Prakash, [15]).

We report about further experiments in similar evolutionary contexts that have justified the grim attitude mentioned above: the veridical perceptions do have a chance for evolutionary survival, should they prove innovative, i. e. able to apply their veridical understanding of the world into an innovation that outperforms the utilitaristic perceptions of the world. However, they are outcompeted by innovative utilitaristic perceptions who take advantage of both the innovation as well as true perception of its utility.

The competition of veridical with utilitaristic perceptions is hence an abstract mathematical model behind the phenomenon of technology valley of death, which was perceived in the documents of EU commission [7], based upon knowledge progress scale introduced by NASA (J. C. Mankins, [21]). It is closely associated with product development (S. K. Markham, S. J. Ward, L. Aiman-Smith, A. I. Kingon, [23]). Our model exhibits technology valley of death as an emergent evolutionary phenomenon in evolutionary environments in which the agents' model of the world is distinct from the true structure of the world, the utilities of resources are non-monotonous in the amount of resources, and the agents are evolving their perceptions (i. e. their model), their decision process, and their actions, gaining evolutionary advantage over those who either perceive less realistically (thus unable to innovate) or less utilitaristically (thus unable to maximize evolutionary utility of the innovations).

This master's thesis is organised into 7 chapters.

We begin by introducing Technology Readiness Level and Valley of Death in Chapter 1. Moreover, a brief overview of research of Valley of Death is given.

In Chapter 2, we establish the language and notation of measure theory. We center our discussion on the measurable spaces, probability measure, Markov kernels and learning space, which remain a key notion in this thesis.

In Chapter 3, we introduce a Minkowski spacetime and prove Noether's Theorem.

Chapter 4 begins by defining the possible relations between the environment and an organism's perceptions. Furthermore, we introduce the mathematical model of perception. We also prove Invention of Symmetry Theorem.

In Chapter 5, we reproduce results of J. T. Mark et al. [22], who develop a simple evolutionary game between perceptual strategies with varying degrees of truthful perceptions. The chapter explores these games with Monte Carlo simulations and finds that natural selection favors perceptions that are tuned to utility rather than truthful representation of the environment.

Furthermore, in Chapter 6 we define a new strategy; critical realist who has a possibility to store surplus resources. Furthermore, we explore a competition between interface strategy and critical realist with storage. We show that under special conditions, critical realist with storage drives interface strategy to extinction.

In Chapter 7, we define another innovative strategy; an interface strategy with storage, and we explore competition between innovative critical realist and innovative interface strategy. We show that innovative interface strategy is the final evolutionary winner.

Chapter 1

Advancement and success of knowledge from basic research to widespread use

Technology is the application of scientific knowledge to solve practical problems, particularly in industry. This chapter will first present the Technology Readiness Levels, a system used to evaluate technology. In second section, an obstacle that appears during the application of science for practical purposes entitled Valley od Death will be presented. Last section consists of a brief overview of research about bridging the mentioned obstacle.

1.1 Technology Readiness Level

The Technology Readiness Level (TRL) is a method used to estimate the maturity of technology. For over 40 years, this method has been implemented in various industries. TRLs help engineers to communicate the progress of development, specify deliverables and manage risks. As TRLs have been embraced in more sectors, the initial domain of TRL application has extended and difficulties in use have developed (K. Tomaschek, A. Olechowski, S. Eppinger, N. Joglekar, [38]).

Any technological development can be related to a TRL between 1 (”we have noticed this occurs, but we don’t understand why and how it happens”) and 9 (“this is used in real-life applications on the market”) (P. Leitner, [19]). The concerete details of the stages between them differ, but the Figure 1.1 shows the 9 TRL levels as seen by the European Commission [7].

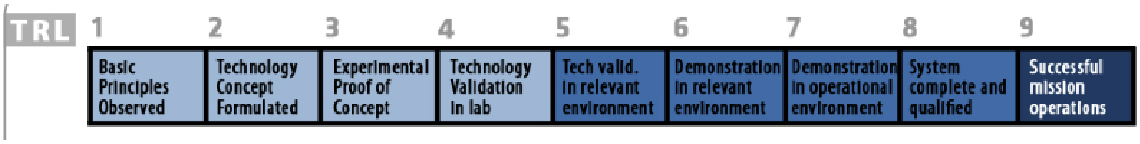


Figure 1.1: Technology Readiness Levels. [Source: European Commission, [7]]

To take an example, research in computer science at universities falls mainly into TRL 2, 3 and 4. It could mature up to 5 in extremely rare cases. But the research never falls into TRL 9 beacuse of the lack of publications to do. It is hard to publish that your tool (which you have already proven conceptually accurate) will function in a appropriate demo setting as well. It is a lot of hard work, with exceptionally small prompt advantage for the academic (P. Leitner [19]).

On the other side, companies are for the most part interested into the products above TRL 7. They may have research departments operating on TRL 6, but because they do not have the correct risk-reward ratio, they are uncommon. In theory, many of the academic concepts in lower TRLs may be true, but their practical usefulness will still be hindered by practical issues (P. Leitner [19]).

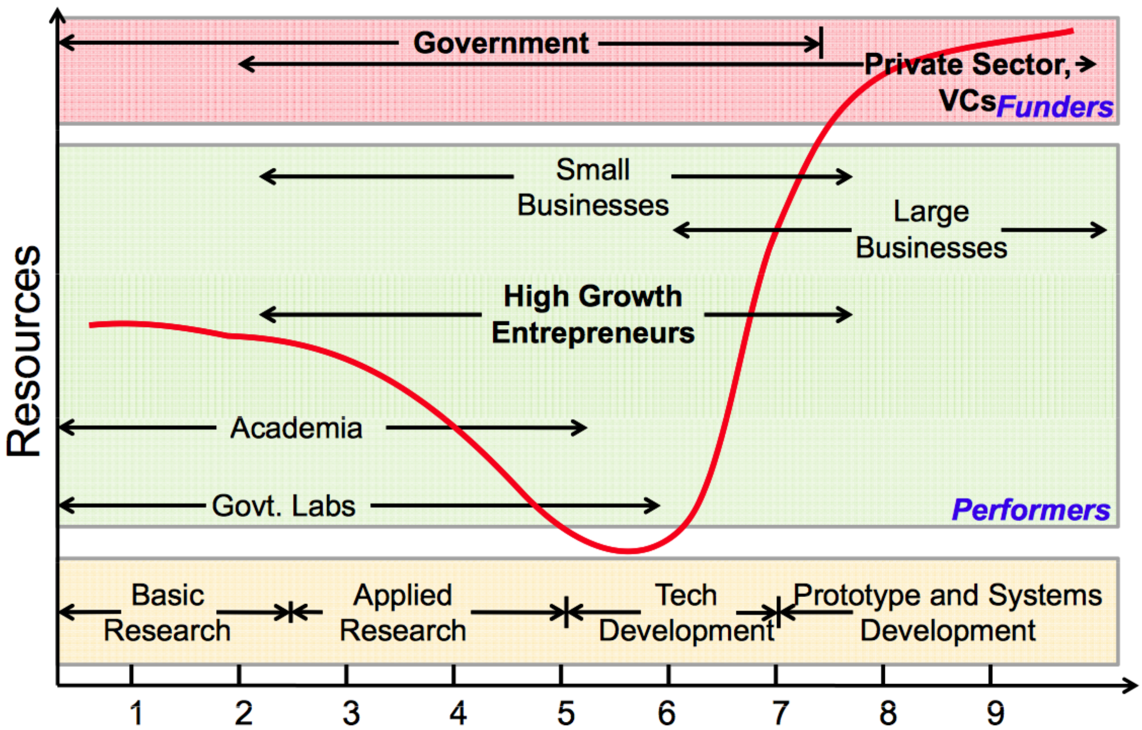


Figure 1.2: Valley of death. [Source: P. Leitner, [19]]

Figure 1.2 shows why there is poor collaboration between industry and academics. Essentially, both sides suppose that the other side will interface with them at the stage at which they usually start/stop. The academic expects that the company will pick up their stuff where research essentially stops and only trivial technical issues are to be solved, which is at TRL 4. The company, on the other hand, assumes that they will receive technology at TRL 7 because there the risks of using new technology have been investigated and contained, and it is shown that the technology can engage in business processes. None of those assumptions will actually happen (P. Leitner [19]).

Obviously, there is a gap that prevents practitioners from selecting the academic ideas, although some of them might be good. This gap is often called the Valley of Death.

1.2 Valley of Death

"Valley of Death" is the metaphor used to describe the gap between academic-based innovations and their market-based commercial applications. In other words, the valley is a crossing between the formal roles, activities, and assets poured into research and the existing formal development roles, activities, processes, and assets that leads toward commercialization. Although technology transfer definitions frequently assume a smooth transition of intellectual property from research facilities to companies that commercially develop the technology, the Valley of Death indicates that the practice is not as smooth. Indeed, this rather grim metaphor implies that academic research is cut off from the outside world in some manner (K. E. Gulbrandsen, [11]).

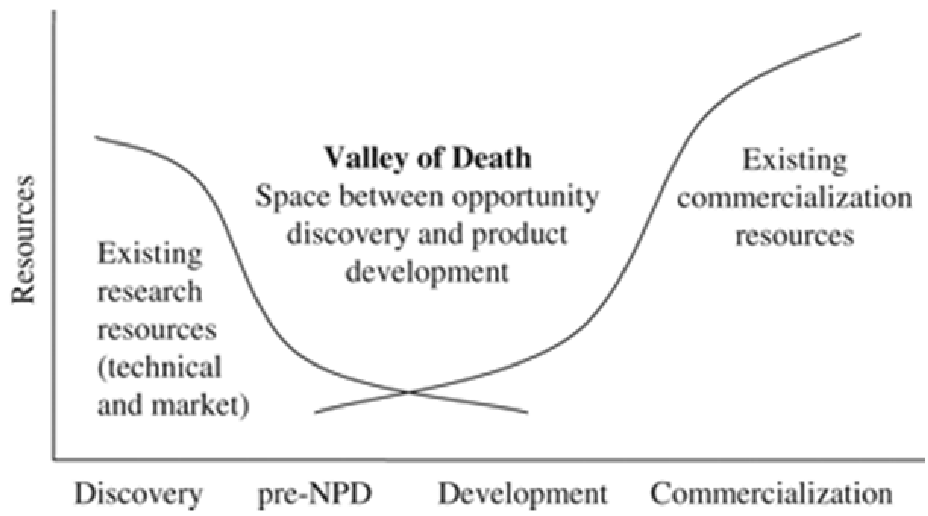


Figure 1.3: Valley of death. [Source: S. K. Markham, S. J. Ward, L. Aiman-Smith, A. I. Kingon, [23]]

Figure 1.3 shows the resource availability on different levels of development. It demonstrates that adequate resources are available during research but then often drop rapidly. On the other side of the valley, resources appear again for developing ideas in order to enter commercialization. As Figure 1.3 suggests, if an idea makes it through the valley from research to development, there is adequate resource availability to take the idea to market. The valley analogy represents a macro view of the structures, processes, people, and resources associated with crossing the gap (S. K. Markham, S. J. Ward, L. Aiman-Smith, A. I. Kingon, [23]).

1.3 A brief overview of Valley of death research

Many university and industry scientists are working together in order to cross the Valley of death as they try to bring basic research to the market. In this section, we provide a brief overview of the current research of Valley of death.

J. Hudson and H. F. Khazragui [16], researchers from the pharmaceutical industry, have examined stress interaction between the various agents throughout the innovation process in the UK, the EU and the USA using several specific examples, suggesting that collaboration between industry and academia is still far from ideal and that the return for academia on its research investment is extremely low.

Furthermore, Swedish researchers T. Lindström and S. Silver [20] conducted the research in a Swedish company within the food science and agriculture. They examined the Valley of Death as an obstacle that appears along the road to commercialization. They also identified the aspects that need to be addressed in order to find out how to work around the barriers to improve the success rate of commercialization.

Moreover, S. K. Markham, S. J. Ward, L. Aiman-Smith, A. I. Kingon, [23] define and explain the front end of product innovation as a discrete segment of development between research and product development. In their research, the Valley of Death is used as a metaphor to describe the absence of resources and expertise in this stage of development. The metaphor suggests that there are more resources on one side of the valley in the form of research expertise and on the other side by commercialization expertise and resources. Within this valley, overlapping roles that move projects from one side of Valley of Death to the other are examined.

Danish researchers S. Debois, T. Hildebrandt, T. Slaats, M. Marquard [6] report on a successful story in academic cooperation with industry: The development of a new technology, all the way from its conception as a potentially interesting academic idea at the University of Copenhagen, to its implementation in a commercial product accessible from Exformatics

A/S proven in operational setting. Thanks to a strategic research project, the ideas were realised into theoretical basis and a proof-of-concept prototype application was validated in the laboratory. However, then they faced the well-known challenge of moving from the laboratory to the real world, i. e. moving from TRL 5 to 7, also known as bridging the "Valley of Death". They identified the main conditions that made it possible to overcome the challenge: The deployment of academic/industrial knowledge-networks; specific types of smaller financing instruments and the "industrial PhD"-model.

The European Commission is also explicitly addressing the Valley of Death in their Horizon 2020 programme. For instance, they have established PPP (Private-Public Partnership) and the EIT (European Institute of Innovation and Technology) to explicitly bring technology through the Valley of Death and make them relevant to society faster (European Commission, [7]).

Chapter 2

Measure theory

Measure theory is the study of measures. A measure on a set is a way for each suitable subset of that set to be assigned a number, which is intuitively interpreted as its size. In this context, a measure is a generalization of the concepts of length, area, and volume. Measure theory was developed by Émile Borel, Henri Lebesgue, Johann Radon, and Maurice Fréchet during the late 19th and early 20th centuries. The main applications of measures are in the foundations of the Lebesgue integral, in Andrey Kolmogorov's axiomatisation of probability theory and in ergodic theory. In this chapter, we define σ-algebras, measurable spaces, measures, probability measure and Markov kernels; structures used in mathematical models we introduce in Chapters 5–7. Furthermore, we define complete metric space to prove Banach Fixed Point Theorem on metric space in order to define learning space and to prove Bounded Learning Theorem on the mentioned space. In last section, we present short intro on group actions in order to prove Invention of Symmetry Theorem in Chapter 4.

2.1 Sets

In general, we can not define a measure of an arbitrary set of subsets of a given set. For that reason, we will restrict the class of sets we consider. The class of sets that we want to use are σ-algebras. General references used in this section are R. F. Bass [3], and M. Papadimitrakis [29].

Definition 2.1. *Let X be a non-empty set. An algebra is a collection $\mathcal{A}$ of subsets of X such that*

1. *$\exists$ at least one $A \subset X$ so that $A \in \mathcal{A}$,*

2. *if* $A \in \mathcal{A}$*, then* $A^c \in \mathcal{A}$*, where* $A^c = X \setminus A$*,*
3. *if* $A_1, \ldots, A_n \in \mathcal{A}$*, then* $\bigcup_{i=1}^{n} A_i \in \mathcal{A}$*.*

$\mathcal{A}$ *is a* σ*-algebra if in addition*

4. *whenever* $A_i \in \mathcal{A}$*,* $\forall\ i \in \mathbb{N}$ *, then* $\bigcup_{i=1}^{+\infty} A_i \in \mathcal{A}$*.*

From de Morgan's laws, a collection of subsets is σ-algebra if it contains $\emptyset$ and is closed under the operations of taking complements and countable unions (or, equivalently, countable intersections).

Proposition 2.2. *Every* σ*-algebra of subsets of* X *contains at least the sets* $\emptyset$ *and* X*, it is closed under countable intersections, under finite intersections and under set-theoretic differences.*

Proof. Let $\mathcal{A}$ be any σ-algebra of subsets of X.

1. Since $\mathcal{A}$ is a σ-algebra, according to 2.1.1, there exists at least one $A \in \mathcal{A}$. Take any $A \in \mathcal{A}$ and consider the sets $A_1 = A$ and $A_n = A^c$, $\forall n \geq 2$. Then $X = A \cup A^c = \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$ and also $X \setminus X = \emptyset \in \mathcal{A}$.
2. Let $A_i \in \mathcal{A}$, $\forall i$. Then $\bigcap_{i=1}^{+\infty} A_i = (\bigcup_{i=1}^{+\infty} A_i^c)^c \in \mathcal{A}$.
3. Let $A_1, \ldots, A_N \in \mathcal{A}$. Consider $A_i = A_N$, $\forall i > N$ and get that $\bigcup_{i=1}^{N} A_i = \bigcup_{i=1}^{+\infty} A_i \in \mathcal{A}$. Then $\bigcap_{i=1}^{N} A_i = (\bigcup_{i=1}^{N} A_i^c)^c \in \mathcal{A}$.
4. Finally, let $A, B \in \mathcal{A}$. Using the result of 3., we get that $A \setminus B = A \cap B^c \in \mathcal{A}$.

□

Example 2.3. *If* X *is a set, then* $\{\emptyset, X\}$ *and* $\mathcal{P}(X)$ *are* σ*-algebras on* X*; they are the smallest and largest* σ*-algebras on* X*, respectively.*

Proposition 2.4. *Let* $\mathcal{A}$ *be a* σ*-algebra of subsets of* X *and consider a finite sequence* $\{A_i\}_{i=1}^{N}$ *or an infinite sequence* $\{A_i\}$ *in* $\mathcal{A}$*. Then there exists a finite sequence* $\{B_i\}_{i=1}^{N}$ *or, respectively, an infinite sequence* $\{B_i\}$ *in* $\mathcal{A}$ *with the properties:*

1. $B_i \subseteq A_i$ *for all* $i = 1, \ldots, N$ *or, respectively, all* $n \in \mathbb{N}$*.*
2. $\bigcup_{i=1}^{N} B_i = \bigcup_{i=1}^{N} A_i$ *or, respectively,* $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$*.*
3. *the* B_n*'s are pairwise disjoint.*

Proof. Trivial, by taking $B_1 = A_1$ and $B_k = A_k \setminus (A_1 \cup \cdots \cup A_{k-1})$, for all $k = 2, \ldots, N$ or, respectively, all $k > 2$. □

Proposition 2.5. *The intersection of any σ-algebras of subsets of the same X is a σ-algebra of subsets of X.*

Proof. Let $\{\mathcal{A}_i\}_{i\in I}$ be any collection of σ-algebras of subsets of X, indexed by an arbitrary non-empty set I of indices, and consider the intersection $\mathcal{A} = \bigcap_{i\in I} \mathcal{A}_i$.

1. Since $\emptyset \in \mathcal{A}_i$, $\forall i \in I$, we get $\emptyset \in \mathcal{A}$ and, hence, $\mathcal{A}$ is non-empty.
2. Let $A \in \mathcal{A}$. Then $\forall i \in I$, $A \in \mathcal{A}_i$ and, since all $\mathcal{A}_i$'s are σ-algebras, $\forall i \in I$, $A^c \in \mathcal{A}_i$. Therefore $A^c \in \mathcal{A}$.
3. Let $A_n \in \mathcal{A}$, $\forall n \in \mathbb{N}$. Then $\forall i \in I$ and all $n \in \mathbb{N}$, $A_n \in \mathcal{A}_i$ and, since all $\mathcal{A}_i$'s are σ-algebras, $\forall i \in I$ we get $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}_i$. Thus $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

□

Definition 2.6. *If $\mathcal{A}$ is any collection of subsets of a set X, then the σ-algebra generated by $\mathcal{A}$ is*

$$\sigma(\mathcal{A}) = \bigcap\{\mathcal{F} \subset \mathcal{P}(X) : \mathcal{A} \subset \mathcal{F} \text{ and } \mathcal{F} \text{ is a } \sigma - \text{algebra.}\} \tag{2.1}$$

This intersection is non-empty, since $\mathcal{P}(X)$ is a σ-algebra that contains $\mathcal{A}$, and an intersection of σ-algebras is a σ-algebra.

Proposition 2.7. *Let $\mathcal{F}$ be any collection of subsets of the non-empty X. Then $\sigma(\mathcal{F})$ is the smallest σ-algebra of subsets of X which includes $\mathcal{F}$. Namely, if $\mathcal{A}$ is any σ-algebra of subsets of X such that $\mathcal{F} \subseteq \mathcal{A}$, then $\sigma(\mathcal{F}) \subseteq \mathcal{A}$.*

Proof. If $\mathcal{A}$ is any σ-algebra of subsets of X such that $\mathcal{F} \subseteq \mathcal{A}$, then $\mathcal{A}$ is one of the σ-algebras whose intersection is denoted $\sigma(F)$. Therefore $\sigma(F) \subseteq \mathcal{A}$. □

2.2 Measurable space

In this section, we will define the space on a σ-algebra. General references used in this section are R. F. Bass [3], and M. Papadimitrakis [29].

Definition 2.8. *The pair $(X, \mathcal{A})$ of a non-empty set X and a σ-algebra $\mathcal{A}$ of subsets of X is called a measurable space.*

The elements of $\mathcal{A}$ are called measurable sets. Several properties of measurable sets are immediate from the definition:

1. The empty set, $\emptyset$, is measurable.

   Proof. Since $\mathcal{A}$ is non-empty, there exists some measurable set A. So, $A \setminus A = \emptyset$ is measurable, by condition 2.2.4. □

2. For A and B any two measurable sets, $A \cap B$, $A \cup B$, and $A \setminus B$ are all measurable.

   Proof. The third is just condition 2.2.4. above. For the second, apply condition 2.1.3. to the sequence $A, B, \emptyset, \emptyset, \ldots$. For the first, note that $A \cup B = A \setminus (A \setminus B)$: Use condition 2.2.4. twice. □

   It follows immediately, by repeated application of these facts, that the measurable sets are closed under taking any finite numbers of intersections, unions, and differences.

3. For $A_1, A_2, \ldots$ measurable, their intersection, $\cap A_i$, is also measurable.

   Proof. First note that we have the following set-theoretic identity: $A_1 \cap A_2 \cap A_3 \cap \cdots = A_1 \setminus (A_1 \setminus A_2) \cup (A_1 \setminus A_3) \cup (A_1 \setminus A_4) \cup \ldots$. Now, on the right, apply condition 2.2.4. to the set differences, and condition 2.1.3 to the union. □

Here are some examples of measurable spaces:

Example 2.9. *1. Let A be any set, and let $\mathcal{A}$ consist only of the empty set $\emptyset$. This is a measurable space.*

2. Let A be any set, and let $\mathcal{A}$ consist of all subsets of A. This is a measurable space.

3. Let $(A, \mathcal{A})$ be any measurable space, and let $K \subset S$ (not necessarily measurable). Let $\mathcal{K}$ denote the collection of all subsets of K that are $\mathcal{A}$-measurable. Then $(K, \mathcal{K})$ is a measurable space.

2.3 Measures

A measure is a countably additive, non-negative, extended real-valued function defined on a σ-algebra. General references used in this section are P. Billingsley [5], and M. Papadimitrakis [29].

Definition 2.10. *Let $(X, \mathcal{A})$ be a measurable space. A function $\mu : \mathcal{A} \to [0, +\infty]$ is called a measure on $(X, \mathcal{A})$ if*

1. $\mu(\emptyset) = 0$,

2. *$\mu(\bigcup_{i=1}^{+\infty} A_i) = \sum_{i=1}^{+\infty} \mu(A_i)$, $\forall$ sequences (A_i) of pairwise disjoint sets which are contained in $\mathcal{A}$. This property is called σ-additivity.*

Definition 2.11. *The triple $(X, \mathcal{A}, \mu)$ of a non-empty set X, a σ-algebra of subsets of X and a measure μ on $\mathcal{A}$ is called a measure space.*

For simplicity, we shall say that μ is a measure on $\mathcal{A}$ or a measure on X.

Proposition 2.12. *Every measure is finitely additive.*

Proof. Let μ be a measure on the σ-algebra $\mathcal{A}$. If $A_1, \dots, A_N \in \mathcal{A}$ are pairwise disjoint, we consider $A_i = \emptyset$, $\forall i > N$, and we get $\mu(\bigcup_{i=1}^{N} A_i) = \mu(\bigcup_{i=1}^{+\infty} A_i) = \sum_{i=1}^{+\infty} \mu(A_i) = \sum_{i=1}^{N} \mu(A_i)$. □

Example 2.13. 1. *The simplest measure is the zero measure which is denoted o and is defined by $o(A) = 0$ for every $A \in \mathcal{A}$.*

2. *Let X be an uncountable set and consider $\mathcal{A} = \{A \subseteq X | A$ is countable or A^c is countable$\}$. We define*

$$\mu(A) = \begin{cases} 0, & \text{if } A \text{ is countable,} \\ 1, & \text{if } A^c \text{ is countable.} \end{cases}$$

Then it is clear that $\mu(\emptyset) = 0$. Let $A_1, A_2, \dots \in \mathcal{A}$ be pairwise disjoint. If all of them are countable, then $\bigcup_{i=1}^{+\infty} A_i$ is also countable and we get $\mu(\bigcup_{i=1}^{+\infty} A_i) = 0 = \sum_{i=1}^{+\infty} \mu(A_i)$. Observe that if one of the A_i's, say A_{i_0}, is uncountable, then for all $i \neq i_0$ we have $A_i \subseteq A_{i_o}^c$ which is countable. Therefore $\mu(A_{i_0}) = 1$ and $\mu(A_i) = 0$ for all $i \neq i_0$. Since $(\bigcup_{i=1}^{+\infty} A_i)^c (\subseteq A_{i_0}^c)$ is countable, we get $\mu(\bigcup_{i=1}^{+\infty} A_i) = 1 = \sum_{i=1}^{+\infty} \mu(A_i)$.

Theorem 2.14. *Let $(X, \mathcal{A}, \mu)$ be a measure space.*

1. *(Monocity) If $A, B \in \mathcal{A}$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$.*
2. *If $A, B \in \mathcal{A}$, $A \subseteq B$ and $\mu(A) < +\infty$, then $\mu(B \setminus A) = \mu(B) - \mu(A)$.*
3. *(σ-subadditivity) If $A_1, A_2, \dots \in \mathcal{A}$, then $\mu(\bigcup_{i=1}^{+\infty} A_i) \leq \sum_{i=1}^{+\infty} \mu(A_i)$.*

Proof. 1. We write $B = A \cup (B \setminus A)$. By finite additivity of μ, $\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A)$.

2. From both sides of $\mu(B) = \mu(A) + \mu(B \setminus A)$ we subtract $\mu(A)$.
3. Using Proposition 2.4 we find $B_1, B_2, \dots \in \mathcal{A}$ which are pairwise disjoint and satisfy $B_i \subseteq A_i$ for all i and $\bigcup_{i=1}^{+\infty} B_i = \bigcup_{i=1}^{+\infty} A_i$. By σ-additivity and monotonicity of μ, we get $\mu(\bigcup_{i=1}^{+\infty} A_i) = \mu(\bigcup_{i=1}^{+\infty} B_i) = \sum_{i=1}^{+\infty} \mu(B_i) \leq \sum_{i=1}^{+\infty} \mu(A_i)$.

□

Definition 2.15. *Let $(X, \mathcal{A}, \mu)$ be a measure space.*

1. *μ is called finite if $\mu(X) < +\infty$.*
2. *μ is called σ-finite if $X = \bigcup_{i=1}^{+\infty} A_i$ is a countable union of measurable sets A_i with finite measure, $\mu(A_i) < +\infty$.*

2.4 Probability measure

Suppose that we have a random experiment with sample space Ω. Intuitively, the probability of an event is a measure of how likely the event is to happen during the experiment. The difference between a probability measure and the general definition of measure is that a probability measure must assign value one to the entire probability space. In this section, the main reference is P. Billingsley [5].

Definition 2.16. *Let Ω be a non-empty set, which is called the sample space. Let $\mathcal{A}$ be a σ-algebra of subsets of Ω, whose elements are called events. P is a probability measure on $\mathcal{A}$ if*

1. *P is a measure on $\mathcal{A}$,*
2. *$P(\Omega) = 1$.*

In particular, P is a finite measure. For any event $A \in \mathcal{A}$, $P(A)$ is called the probability of A.

Since $\mathcal{A}$ is an algebra, that is, $\Omega \in \mathcal{A}$, the operation of taking complement is defined in $\mathcal{A}$, that is, if $A \in \mathcal{A}$ then the complement $A^c := \Omega \setminus A$ is also in $\mathcal{A}$. The event A^c is opposite to A and

$$P(A^c) = P(\Omega \setminus A) = P(\Omega) - P(A) = 1 - P(A). \tag{2.2}$$

Definition 2.17. *Let Ω be a sample space. If $\mathcal{A}$ is a σ-algebra in Ω and P is a probability measure on $\mathcal{A}$, the triple $(\Omega, \mathcal{A}, P)$ is called a probability measure space, or simply a probability space.*

Example 2.18. *Let $\Omega = \{1, 2, ..., N\}$ be a finite set, and $\mathcal{A}$ be the set of all subsets of Ω. Given N non-negative numbers p_i such that $\sum_{i=1}^{N} p_i = 1$, define P by*

$$P(A) = \sum_{i \in A} p_i. \tag{2.3}$$

From the condition (2.1), we have $P(\Omega) = 1$.
For instance, in the case $N = 2$, $\mathcal{A}$ *consists of the sets* $\emptyset, \{1\}, \{2\}, \{1,2\}$. *Given two non-negative numbers* p *and* q *such that* $p + q = 1$, *set* $P(\{1\}) = p$, $P(\{2\}) = q$, *while* $P(\emptyset) = 0$ *and* $P(\{1,2\}) = 1$.
Furthermore, we can generalize this example to the case when Ω *is a countable set, for example,* $\Omega = \mathbb{N}$. *Given a sequence* $\{p_i\}_{i=1}^{+\infty}$ *of non-negative numbers* p_i *such that* $\sum_{i=1}^{+\infty} p_i = 1$, *define for any set* $A \subset \Omega$ *its probability by (2.1). Measure* P *constructed by (2.1) is called a discrete probability measure, and the corresponding space* $(\Omega, \mathcal{A}, P)$ *is called a discrete probability space.*

2.5 Complete metric space

Another space we use in this chapter is complete metric space. Therefore, in this section, we present a metric space, as well as a complete metric space. We use mentioned space in Section 2.6 in order to prove Banach Fixed Point Theorem. Main references for this section are A. Gonzalez [10], M. Searcoid [36] and S. Shirali, H. L. Vasudeva [37].

Let begin with the definition of a metric, which is, roughly speaking, a rule to measure the distance between two elements from the same set:

Definition 2.19. *Let* X *be a set. A metric on the set* X *is a function* $d : X \times X \to [0, \infty)$ *such that the following conditions are satisfied for all* $x, y, z \in X$:

1. *(Non-negativeness)* $d(x, y) \geq 0$,
2. *(Identification)* $d(x, y) = 0 \Leftrightarrow x = y$,
3. *(Symmetry)* $d(x, y) = d(y, x)$,
4. *(Triangle inequality)* $d(x, z) \leq d(x, y) + d(y, z)$.

Definition 2.20. *Given a set* X *and a metric* $d : X \times X \to \mathbb{R}$ *on the set* X, *we say that the pair* (X, d) *is a metric space.*

The main purpose of this section is to give conditions for a space to be complete. First of all, let recall the notion of Cauchy sequence.

Definition 2.21. *Let* (X, d) *be a metric space. A sequence* $\{x_n\}_{n \in \mathbb{N}}$ *in* X *is a Cauchy sequence if, for all* $\varepsilon > 0$, *there exists some* $M \in \mathbb{N}$ *such that*

$$d(x_n, x_m) < \varepsilon, \text{ for all } n, m \geq M. \tag{2.4}$$

Now let define the complete metric space:

Definition 2.22. *Let (X, d) be a metric space. A metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent.*

Proposition 2.23. *In a complete metric space, a sequence is convergent if and only if it is Cauchy.*

Proof. First, let assume that (x_n) is a sequence which converges to x. Let $\varepsilon > 0$ be given. Then there is an $N \in \mathbb{N}$ such that $d(x_n, x) < \frac{\varepsilon}{2}$, for all $n \geq N$. Let $m, n \in N$ be such that $m \geq N$, $n \geq N$. Then

$$d(x_m, x_n) \leq d(x_m, x) + d(x_n, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon.$$

It follows that (x_n) is a Cauchy sequence.
On the other side, let assume that (x_n) is a Cauchy sequence in a complete metric space. Since metric space is complete, it follows by definition that (x_n) is a convergent sequence in the given complete metric space. □

Proposition 2.24. *Let (X, d) be a complete metric space and $A \subseteq X$. Then (A, d) is complete if and only if A is closed.*

Proof. Assume first that A is closed and $\{x_n\}$ is a Cauchy sequence in A, then $\{x_n\}$ is a Cauchy sequence in X, so it converges to $x_0 \in X$. Since A is closed, $x_0 \in A$. This shows that A is complete.
Now let assume that A is complete, but it is not closed. Then there exists a convergent sequence in A whose limit does not belong to A. Since it converges, the sequence is Cauchy and has a limit in X, but it doesn't have a limit in A, so A is not complete, which is a contradiction. □

2.6 Banach Fixed Point Theorem

In this section, we present Banach Fixed Point Theorem (also known as the contraction mapping theorem), which is an important tool in the theory of metric spaces. It's importance is in it guaranteeing the uniqueness and existence of fixed points of certain self-maps of metric spaces. We use this theorem in Section 2.8. Main reference for this section is S. Banach [2].

Definition 2.25. *Let (X, d) be a complete metric space. A map $C\colon X \to X$ is called a contraction mapping on X if there exists $q \in [0, 1)$ such that*

$$d(C(x), C(y)) \leq q d(x, y) \tag{2.5}$$

for all $x, y \in X$.

Theorem 2.26. *(Banach Fixed Point Theorem) Let (X, d) be a non-empty complete metric space and let $C\colon X \to X$ be a contraction mapping. Then there exists a unique $x^* \in X$ such that*

$$C(x^*) = x^*. \tag{2.6}$$

Remark. A method to find x^*: begin with an arbitrary element $x_0 \in X$ and define a sequence x_n by $x_n = C(x_{n-1})$. Then $x_n \longrightarrow x^*$.

Proof. Let $x_0 \in X$ be arbitrary. We now define a sequence x_n by setting $x_n = C(x_{n-1})$. Then by induction on n and using the fact that T is a contraction mapping, we have the inequality:

$$d(x_{n+1}, x_n) \leq q^n d(x_1, x_0), \text{ for all } n \in N.$$

Now let show that x_n is a Cauchy sequence. In particular, let $m, n \in \mathbb{N}$ such that $m > n$:

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\ &\leq q^{m-1} d(x_1, x_0) + q^{m-2} d(x_1, x_0) + \cdots + q^n d(x_1, x_0) \\ &= q^n d(x_1, x_0) \textstyle\sum_{k=0}^{m-n-1} q^k \\ &\leq q^n d(x_1, x_0) \textstyle\sum_{k=0}^{\infty} q^k \\ &= q^n d(x_1, x_0) \left(\tfrac{1}{1-q}\right) \end{aligned}$$

Let $\varepsilon > 0$ be arbitrary, since $q \in [0, 1)$, we can find a large $N \in \mathbb{N}$ so that

$$q^N < \tfrac{\varepsilon(1-q)}{d(x_1, x_0)}.$$

Therefore, by choosing m and n greater than N we have:

$$d(x_m, x_n) \leq q^n d(x_1, x_0) \left(\tfrac{1}{1-q}\right) < \left(\tfrac{\varepsilon(1-q)}{d(x_1, x_0)}\right) d(x_1, x_0) \left(\tfrac{1}{1-q}\right) = \varepsilon.$$

Thus, the sequence x_n is Cauchy. Since (X, d) is a complete metric space, the sequence has a limit $x^* \in X$. Furthermore, x^* must be a fixed point of C:

$$x^* = \lim_{n\to\infty} x_n = \lim_{n\to\infty} C(x_{n-1}) = C\left(\lim_{n\to\infty} x_{n-1}\right) = C(x^*).$$

As a contraction mapping, C is continuous, so we could bring the limit inside C.
Lastly, let show the uniqueness of a fixed point. C cannot have more than one fixed point in (X, d), since any pair of distinct fixed points p_1 and p_2 would contradict the contraction of C:

$$0 < d(C(p_1), C(p_2)) = d(p_1, p_2) > qd(p_1, p_2).$$

□

2.7 Markov kernel

Another mathematical object that we use is the Markov kernel, also called a stochastic kernel, transition probabilities or a regular conditional probability distribution. It plays an important role in probability theory and mathematical statistics. In this section, we recall some useful facts about Markov kernels. Main references for this section are A. G. Nogales [25], and P. Panangaden [28].

In the next, $(\Omega_1, \mathcal{A}_1)$, $(\Omega_2, \mathcal{A}_2)$,... denote measurable spaces. A random variable is a map $X : (\Omega_1, \mathcal{A}_1) \to (\Omega_2, \mathcal{A}_2)$ such that $X^{-1}(A_2) \in \mathcal{A}_1$, $\forall A_2 \in \mathcal{A}_2$. Its probability distribution (or, simply, distribution) P^X with respect to a probability measure P on $\mathcal{A}_1$ is the image measure of P by X, i. e. the probability measure on $\mathcal{A}_2$ defined by $P^X(A_2) := P(X^{-1}(A_2))$.

Definition 2.27. *Let $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ be measurable spaces. A Markov kernel $M : (\Omega_1, \mathcal{A}_1) \to (\Omega_2, \mathcal{A}_2)$ is a map $M : (\Omega_1, \mathcal{A}_2) \to [0, 1]$ satisfying the following conditions:*

1. *$\forall \omega \in \Omega_1$, $M(\omega, \cdot)$ is a probability measure on $\mathcal{A}_2$,*
2. *$\forall A_2 \in \mathcal{A}_2$, $M(\cdot, A_2)$ is $\mathcal{A}_1$-measurable.*

Remark. Given two random variables $X_i : (\Omega, \mathcal{A}, P) \to (\Omega_i, \mathcal{A}_i)$, $i = 1, 2$, the conditional distribution of X_2 given X_1, when it exists, is a Markov kernel $M : (\Omega_1, \mathcal{A}_1) \to (\Omega_2, \mathcal{A}_2)$ such that $P(X_1 \in A_1, X_2 \in A_2) = \int_{A_1} M(\omega_1, A_2) dP^{X_1}(\omega_1)$, for all $A_1 \in \mathcal{A}_1$, and $A_2 \in \mathcal{A}_2$. We write $P^{X_2|X_1=\omega_1}(A_2) := M(\omega_1, A_2)$. Reciprocally, every Markov kernel is a conditional distribution; namely, given a Markov kernel $M_1 : (\Omega, \mathcal{A}, P) \to (\Omega_1, \mathcal{A}_1)$, it is easily checked that

$$M_1(\omega, A_1) = (P \otimes M_1)^{\pi_1|\pi=\omega}(A_1), \tag{2.7}$$

where $\pi : \Omega \times \Omega_1 \to \Omega$ and $\pi_1 : \Omega \times \Omega_1 \to \Omega_1$ are the coordinatewise projections and $P \otimes M_1$ stands for the only probability measure on the product space $(\Omega \times \Omega_1, A \times A_1)$ such that $(P \otimes M_1)(A \times A_1) = \int_A M_1(\omega, A_1) dP(\omega)$ for all $A \in \mathcal{A}$ and $A_1 \in \mathcal{A}_1$.

Definition 2.28. *The image (or probability distribution) of a Markov kernel* $M_1 : (\Omega, \mathcal{A}, P) \to (\Omega_1, \mathcal{A}_1)$ *on a probability space is the probability measure* P^{M_1} *on* $\mathcal{A}_1$ *defined by*

$$P^{M_1}(A_1) := \int_\Omega M_1(\omega, A_1) dP(\omega). \tag{2.8}$$

Remark. Note that

$$P^{M_1} = (P \otimes M_1)^{\pi_1} \tag{2.9}$$

where $\pi_1 : \Omega \times \Omega_1 \to \Omega_1$ denotes the coordinatewise projection onto Ω_1. So, if $f : (\Omega_1, \mathcal{A}_1) \to \mathbb{R}$ is a nonnegative or P^{M_1}-integrable function, then

$$\int_{\Omega_1} f(\omega_1) dP^{M_1}(\omega_1) = \int_\Omega \int_{\Omega_1} f(\omega_1) M_1(\omega, d\omega_1) dP(\omega) = \int_{\Omega \times \Omega_1} f(\omega_1) d(P \otimes M_1)(\omega, \omega_1). \tag{2.10}$$

Definition 2.29. *The composition of two Markov kernels* $M_1 : (\Omega_1, \mathcal{A}_1) \to (\Omega_2, \mathcal{A}_2)$ *and* $M_2 : (\Omega_2, \mathcal{A}_2) \to (\Omega_3, \mathcal{A}_3)$ *is defined as the Markov kernel* $M_2 M_1 : (\Omega_1, \mathcal{A}_1) \to (\Omega_3, \mathcal{A}_3)$ *given by*

$$M_2 M_1(\omega_1, A_3) = \int_{\Omega_2} M_2(\omega_2, A_3) M_1(\omega_1, d\omega_2). \tag{2.11}$$

Definition 2.30. *Let* $X_1 : (\Omega, \mathcal{A}) \to (\Omega_1, \mathcal{A}_1)$ *be a random variable and* $M_1 : (\Omega_1, \mathcal{A}_1) \to (\Omega_1', \mathcal{A}_1')$ *a Markov kernel. A new Markov kernel* $M_1 X_1 : (\Omega, \mathcal{A}) \to (\Omega_1', \mathcal{A}_1')$ *is defined by means of*

$$M_1 X_1(\omega, A_1') := M_1(X_1(\omega), A_1'). \tag{2.12}$$

Remark. When M_{X_1} is the Markov kernel corresponding to the random variable X_1, we have that $M_1 X_1 = M_1 M_{X_1}$.

2.8 Learning space

In addition to previous section, here we will define the space of all Markov kernels and a metric on it. Furthermore, we will present an application of a Banach Fixed Point Theorem on the mentioned space.

Definition 2.31. *Assume that* $\mathcal{A}$ *and* $\mathcal{B}$ *are* $\sigma-$*algebras and that* $(X, \mathcal{A})$ *and* $(Y, \mathcal{B})$ *are measurable spaces. The space of all Markov kernels from* $(X, \mathcal{A})$ *to* $(Y, \mathcal{B})$ *is called pre-learning space* $L(X, \mathcal{A}, Y, \mathcal{B})$.

Proposition 2.32. *Let $L(X, \mathcal{A}, Y, \mathcal{B})$ be the pre-learning space and $K, K' \in L(X, \mathcal{A}, Y, \mathcal{B})$. Define $d : L(X, \mathcal{A}, Y, \mathcal{B}) \times L(X, \mathcal{A}, Y, \mathcal{B}) \to \mathbb{R}$ by:*

$$d(K, K') = \max_{x,B} |K(x, B) - K'(x, B)|. \tag{2.13}$$

Then d is a metric on $L(X, \mathcal{A}, Y, \mathcal{B})$.

Proof. Let us show that 2.11 is actually a metric. For any Markov kernels $K, K' \in L(X, \mathcal{A}, Y, \mathcal{B})$, $\forall(x, B) \in X \times \mathcal{B} : |K(x, B) - K'(x, B)| \geq 0$, so the maximum $d(K, K')$ of these is larger than or equal to zero as well. Also, $\forall(x, B) \in X \times \mathcal{B} : d(K, K') = 0$ if and only if $|K(x, B) - K'(x, B)| = 0$, which means that $K = K'$. Therefore, d satisfies the first requirement in the definition of a metric.
For any $K, K' \in L(X, \mathcal{A}, Y, \mathcal{B})$ and for all $(x, B) \in X \times \mathcal{B}$, since $|K(x, B) - K'(x, B)| = |K'(x, B) - K(x, B)|$, the maximum of $|K(x, B) - K'(x, B)|$ is the same as the maximum of $|K'(x, B) - K(x, B)|$, so $d(K, K') = d(K', K)$, which is the second requirement.
Finally, suppose $K, K', K'' \in L(X, \mathcal{A}, Y, \mathcal{B})$. Then $\forall(x, B) \in X \times \mathcal{B}$: $|K(x, B) - K''(x, B)| = |K(x, B) - K'(x, B) + K'(x, B) - K''(x, B)| \leq |K(x, B) - K'(x, B)| + |K'(x, B) - K''(x, B)|$ by the triangle inequality for the absolute value function on $\mathbb{R}$. Furthermore, $|K(x, B) - K'(x, B)| + |K'(x, B) - K''(x, B)| \leq \max_{x,B} |K(x, B) - K'(x, B)| + \max_{x,B} |K'(x, B) - K''(x, B)| = D(K, K') + d(K', K'')$. This is the triangle inequality for d, so we conclude that d is a metric on $L(X, \mathcal{A}, Y, \mathcal{B})$. □

Definition 2.33. *Let $L(X, \mathcal{A}, Y, \mathcal{B})$ be the pre-learning space. $L(X, \mathcal{A}, Y, \mathcal{B})$ is a learning space if metric d is complete.*

Definition 2.34. *Let $L(X, \mathcal{A}, Y, \mathcal{B})$ be the learning space. $\mathcal{L} : L(X, \mathcal{A}, Y, \mathcal{B}) \to L(X, \mathcal{A}, Y, \mathcal{B})$ is a learning operator if $\mathcal{L}$ is a contraction (see Definition 2.24).*

Theorem 2.35. *Suppose that $\mathcal{L}$ is a learning operator over a learning space $L(X, \mathcal{A}, Y, \mathcal{B})$. Then there exists a unique Markov kernel K in $L(X, \mathcal{A}, Y, \mathcal{B})$ such that iterative applications of $\mathcal{L}$ converge to K.*

Proof. The existence and uniqueness of such Markov kernel can be shown directly with the Banach Fixed Point Theorem (2.20), since we have a suitable choice of the metric and since the map $\mathcal{L}$ a contraction. □

Theorem 2.36. *Let $L(X, \mathcal{A}, Y, \mathcal{B})$ be a learning space and $\mathcal{L}$ a learning operator on $L(X, \mathcal{A}, Y, \mathcal{B})$. Then there exists a Markov kernel which is maximal element of what $\mathcal{L}$ type learning can teach an agent.*

Proof. The proof follows directly from the definition of a metric (2.25) and Theorem 2.28. □

2.9 Group actions

In this section, we present a brief introduction to the Group theory, with emphasis on group actions. This section is important, because it is used in the Invention of Symmetry Theorem, proved in Chapter 4. Main reference for this section is M. Reeder [33].

Definition 2.37. *A group G is a set equipped with a function, $\cdot : G \times G \to G$, assigning to each pair (a, b) of elements of G another element $a \cdot b \in G$, satisfying the following three axioms:*

1. *(associativity) $a \cdot (b \cdot c) = (a \cdot b) \cdot c, \forall a, b, c \in G$,*
2. *(identity element) There $\exists$ an element $e \in G$ such that $a \cdot e = e \cdot a = a, \forall a \in G$,*
3. *(inverse) $\forall a \in G$, there $\exists a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$, where $e \in G$ is an identity element.*

Definition 2.38. *Let G an G' be groups. A homomorphism of groups G, G' is a function $f : G \to G'$ satisfying*

$$f(g_1 g_2) = f(g_1) f(g_2), \forall g_1, g_2 \in G. \tag{2.14}$$

Accordingly, $f(1_G) = 1'_G$ and $f(g^{-1}) = f(g)^{-1}$, $\forall g \in G$.

Definition 2.39. *The kernel of a homomorphism $f : G \to G'$ is the subset of G defined by*

$$Ker f = \{g \in G : f(g) = 1_{G'}\}. \tag{2.15}$$

Definition 2.40. *An isomorphism $f : G \to G'$ is a bijective group homomorphism.*

Definition 2.41. *A left coset of a subgroup $H < G$ is a subset of G given by*

$$gH = \{gh : h \in H\}. \tag{2.16}$$

Two left cosets are either equal or disjoint; we have $gH = g'H \Leftrightarrow g^{-1}g' \in H$. The set of left cosets of $H \in G$ is denoted G/H, and is called the quotient of G by H. Analogously, we define a right coset $H \setminus G$ as

$$Hg = \{hg : h \in H\}. \tag{2.17}$$

Definition 2.42. *Let X be a set. The symmetric group on X is the group S_X of bijections $\sigma : X \to X$, where the group operation is composition of functions:*

$$\sigma\tau(x) = \tau\sigma(x). \tag{2.18}$$

The elements of S_X are called permutations.

Definition 2.43. *Let X be a set and G a group on X. A G-action on X is a homomorphism $\phi : G \to S_x$ from G into the group S_X of permutations of X. If ϕ is a G-action on X, we say that G acts on X.*

The pair (X, ϕ) is called a G-set or a G-action.

Definition 2.44. *Let (X, ϕ) and (Y, ψ) be two G-sets. A function $f : X \to Y$ is called G-equivariant if*

$$f(\phi(g)x) = \psi(g)f(x), \forall g \in G \text{ and } x \in X. \tag{2.19}$$

Definition 2.45. *We say that (X, ϕ) and (Y, ψ) are equivalent G-sets if there exists a G-equivariant bijection $f : X \to Y$.*

Definition 2.46. *Let X be a set. The stabilizer or fixer of a point $x \in X$ is the subgroup of G given by*

$$G_x = \{g \in G : g \cdot x = x\} \leq G. \tag{2.20}$$

Definition 2.47. *Let X be a set. The orbit of an element $x \in X$ is the subset of X given by*

$$G \cdot x = \{g \cdot x : g \in G\}. \tag{2.21}$$

Orbits are equivalence classes under the equivalence relation $x \sim y$ if $y = g \cdot x$ for some $g \in G$. Since two orbits are either equal or disjoint; the orbits form a partition of X. We write $G\ X$ for the set of orbits.

Definition 2.48. *A G-set on X is transitive if for all $x, y \in X$ there exists $g \in G$ such that*

$$g \cdot x = y. \tag{2.22}$$

Equivalently, the action is transitive if and only if X consists of a single G-orbit. For a general group action, each orbit is a transitive G-set.

Theorem 2.49. *Let X be a set and G a group on X. If a group G acts on a set X, then for each $x \in X$ we have a G-equivariant bijection $f : G/G_x \to G \cdot x$, given by*

$$f(gG_x) = g \cdot x. \tag{2.23}$$

Proof. Because $\forall h \in G_x$ we have $(gh) \cdot x = g \cdot (h \cdot x) = g \cdot x$, the function f is well defined. Next we show that f is a bijection. First, let show that f is an injection:

If $g \cdot x = g' \cdot x$, then $g^{-1}g' \cdot x = x$, so $g^{-1}g' \in G_x$, which means that $gG_x = g'G_x$.

Furthermore, by the definition of the orbit $G \cdot x$, the function f is surjective.
It is left to show that f is G-equivariant. For all $g, g' \in G$ and $x \in X$ we have

$$f(g \cdot g'G_x) = f(gg'G_x) = (gg') \cdot x = g \cdot (g' \cdot x) = g \cdot f(g'G_x).$$

Therefore, f is G-equivariant. $\square$

Chapter 3

Symmetry of Minkowski spacetime

The mathematics gives us an insight into how space and time are inseparably intertwined. The most natural way to see this is in a representation of the world with four dimensions, three spatial and one temporal. In this chapter, we introduce a Minkowski spacetime. Furthermore, we prove the Noether's Theorem. In the last section, we present symmetries of Minkowski spacetime.

3.1 Minkowski spacetime

Minkowski spacetime (or Minkowski space) is a setting which combines three-dimensional Euclidean space and time into a four-dimensional manifold. There, the spacetime interval between any two events is independent of the inertial frame of reference in which they are recorded. It was initially developed for Maxwell's equations of electromagnetism by mathematician Hermann Minkowski. Despite that, the mathematical structure of Minkowski spacetime was shown to be an obvious consequence of the special relativity axioms (L. D. Landau, [18]).

The Minkowski spacetime models the universe we live in, but it does not model the world we percieve. Although we exist in Minkowski spacetime, we percieve the world as three-dimensional Euclidean space with time as a parameter. These two spaces are closely related, but their mathematical structure is quite different. In Euclidean space, the distance between two points is measured by Euclidean metric, which uses the Pythagorean Theorem. Furthermore, Newtonian physics also uses the Euclidean metric for space, but with time as a parameter. On the other side, relativity uses the Minkowski metric to measure the distance between two events in spacetime (E. Nešović, E. B. K. Öztürk, U. Öztürk, [24]).

In general, an event is defined as a specific time and space location, for example, the place where you are currently sitting. As already mentioned, spacetime is a manifold, defined as a collection of all events. You may think of a manifold as a set of points with an approximate local vector space structure. For instance, manifold is a smooth curve, which is a set of points, where at each point, the curve is approximately the tangent line. At each point on a smooth curve, there is an approximate local vector space structure (tangent line = one-dimensional vector space). Moreover, a coordinate system is the function that maps a point on a manifold to a vector in a vector space (E. Nešović, E. B. K. Öztürk, U. Öztürk [24]).

Definition 3.1. *(P. A. Quang, [31]) A metric tensor g on a smooth manifold M is a symmetric nondegenerate $(0, 2)$ tensor field on M. A smooth manifold M equipped with a metric tensor g is called a Riemannian manifold.*

We use $\langle x, y\rangle$ as an alternative notation for g, so that

$$g(x, y) = \langle x, y\rangle, \tag{3.1}$$

in which x, y are tangent vectors.

Definition 3.2. *(P. A. Quang, [31]) Minkowski spacetime (M, η) is the manifold $\mathbb{R}^4_1$ endowed with the Minkowski inner product $\langle, \rangle$. For a coordinate system (x^0, x^1, x^2, x^3), one gets the components of the metric tensor g*

$$g_{\mu\nu} = \langle \frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial y^\nu} \rangle := \eta_{\mu\nu}, \tag{3.2}$$

where $\mu, \nu = 0, 1, 2, 3$.

In four-dimensional space, the coordinates of an event (ct, x, y, z) are the components of a four-dimensional radius vector denoted by (x^0, x^1, x^2, x^3). Furthermore, origin, which is a tangent vector in a tangent space at a fixed event can be expressed as

$$x = \sum_\mu x^\mu \frac{\partial}{\partial x^\mu}. \tag{3.3}$$

For two tangent vectors x, y in M, their Minkowski inner product is then

$$g(x, y) = \langle x, y\rangle := -x^0y^0 + x^1y^1 + x^2y^2 + x^3y^3 = \sum_{\mu,\nu} \eta_{\mu\nu}x^\mu y^\nu. \tag{3.4}$$

An orthonormal basis for this inner product is then

$$\Big\{\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\Big\}. \tag{3.5}$$

We have presented Minkowski inner product. Now we present some basic properties of the Minkowski spacetime. (P. A. Quang, [31])

Definition 3.3. *The length of a vector $x \in \mathbb{R}^4_1$ has the form*

$$|x| = \sqrt{|\langle x, x\rangle|}. \tag{3.6}$$

Definition 3.4. *A vector $x \in \mathbb{R}^4_1$ is called:*

1. *timelike, if the inner product of x with itself is negative:*

$$\langle x, x\rangle < 0, \tag{3.7}$$

2. *spacelike, if the inner product of x with itself is positive:*

$$\langle x, x\rangle > 0, \tag{3.8}$$

3. *lightike, if the inner product of x with itself vanishes:*

$$\langle x, x\rangle = 0, \tag{3.9}$$

3.2 Noether's Theorem

In essence, Noether's Theorem states that when an action has a symmetry, it is possible to derive a conserved quantity. In order to prove this theorem, we first have to define a symmetry and a conserved quantity (W. S. Gang, [8]).

Definition 3.5. *(D. Wheeler, [41]) Let C be an arbitrary compact region of spacetime. The action*

$$S[x(t)] = \int_C L(x^i, \dot{x}^i, t)dt \tag{3.10}$$

is extremal, if $x^i(t)$ satisfies the Euler-Lagrange equation,

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = 0. \tag{3.11}$$

This condition guarantees that δS vanishes for all variations, $x^i(t) \to x^i(t) + \delta x^i(t)$ which vanish at the endpoints of the motion. Let $x^i(t)$ be a solution to the Euler-Lagrange equation. Then a function of $x^i(t)$ and its time derivatives,

$$f(x^i(t), \dot{x}^i(t), \dots) \tag{3.12}$$

is conserved, if it is constant along the paths of motion

$$\left.\frac{df}{dt}\right|_{x^i(t)} = 0. \tag{3.13}$$

Sometimes it is the case that δS vanishes for certain limited variations of the path without imposing any condition at all. When this happens, we say that S has a symmetry.

Definition 3.6. *A symmetry of an action functional $S[x]$ is a transformation of the path, $x^i(t) \to \lambda^i(x^j(t), t)$ that leaves the action invariant,*

$$S[x^i(t)] = S[\lambda^i(x^j(t), t)] \tag{3.14}$$

regardless of the path of motion $x^i(t)$. In particular, when $\lambda^i(x)$ represents a continuous transformation of x^1, we may expand the transformation infinitesimally, so that

$$x^i \to x'^i = x^i + \varepsilon^i(x) \tag{3.15}$$

$$\delta x^i = x'^i - x^i = \varepsilon^i(x). \tag{3.16}$$

Since the infinitesimal transformation must leave $S[x]$ invariant, we have

$$\delta_\varepsilon S = S[x^i + \varepsilon^i(x)] - S[x^i] = 0 \tag{3.17}$$

whether $x(t)$ satisfies the field equations or not. If an infinitesimal transformation is a symmetry, several infinitesimal transformations can be arbitrarily applied to recover the invariance of S under finite transformations. $\lambda(x)$ is here a specific function of the coordinates. There is not placing any new demand on the action, just noticing that particular transformations do not change it.

Theorem 3.7 (Noether's Theorem)**.** *(D. Wheeler, [41]) Suppose an action dependent on N independent functions $x^i(t)$, $i = 1, 2, \ldots, N$, has a symmetry so that it is invariant under*

$$\delta_\varepsilon x^i = x'^i - x^i = \varepsilon^i(x), \tag{3.18}$$

where $\varepsilon^i(x)$ are fixed functions of $x^i(t)$. We carefully distinguish between the symmetry variation δ_ε and a general variation δ. Then the quantity

$$I = \frac{\partial L(x(\lambda))}{\partial \dot{x}^i}\varepsilon^i(x) \tag{3.19}$$

is conserved.

Proof. (D. Wheeler, [41]) According to Definition 3.6., the existence of a symmetry means that

$$\begin{aligned} 0 &= \delta_\varepsilon S[x(t)] \\ &= \textstyle\sum_{i=1}^{N} \int_{t_1}^{t_2} \left(\frac{\partial L(x(t))}{\partial x^i} \varepsilon^i(x) + \left(\frac{\partial L(x(t))}{\partial \dot{x}^i_{(n)}} \right) \frac{d\varepsilon^i(x)}{dt} \right) dt \end{aligned}$$

Notice that δS vanishes identically due to the symmetry of the action. There was no equation of motion used.

Now let integrate the second term by parts. We get

$$\begin{aligned} 0 &= \textstyle\int \left(\frac{\partial L}{\partial x^i} \varepsilon^i(x) + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^i} \varepsilon^i(x) \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^i} \right) \varepsilon^i(x) \right) dt \\ &= \frac{\partial L}{\partial \dot{x}^i} \varepsilon^i(x) \Big|_{t_1}^{t_2} + \textstyle\int \left(\frac{\partial L}{\partial x^i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^i} \right) \right) \varepsilon^i(x) dt. \end{aligned}$$

This expression vanishes for every path.

Now suppose that classical path $x^i(t)$ satisfies the Euler-Lagrange equation (3.11). Then for that path, the integrand vanishes and we get

$$\begin{aligned} 0 &= \delta S[x] \\ &= \frac{\partial L}{\partial \dot{x}^i} \varepsilon^i(x(t)) \Big|_{t_1}^{t_2} = I(t_2) - I(t_1) \end{aligned}$$

for any two end times, t_1, t_2.
Therefore,

$$\begin{aligned} \frac{dI}{dt} &= 0 \\ \text{and } I &= \frac{\partial L(x, \dot{x})}{\partial \dot{x}^i} \varepsilon^i \end{aligned}$$

is a constant of the motion. □

3.3 Symmetries

A metric has a symmetry when there is a coordinate transformation that does not change the components of a metric. Let show that the Minkowski spacetime metric is invariant under ten symmetry transformations. General reference for this section is D. Wheeler [40].

Let start with a definition of a covariant derivative:

Definition 3.8. *The covariant derivative of a covariant tensor X_a is*

$$X_{a;b} = \frac{\partial X_a}{\partial x^b} - \Gamma^k_{ab} X_k, \tag{3.20}$$

where Γ^k_{ab} is a second kind of the Christoffel symbol, which is a tensor-like object derived from a Riemannian metric.

Now let define a Killing equation:

Definition 3.9. *A Killing vector field is a type of a symmetry defined as a smooth vector field that preserves the metric tensor. This is usually called a Killing equation and written as:*

$$X_{a;b} + X_{b;a} = 0, \tag{3.21}$$

where $X_{a;b}$ is a covariant derivative.

Given the metric, we search for all solutions to the Killing equation (3.21). Solutions, if they exist, represent symmetry directions of the spacetime, i. e. directions in which the metric is unchanging.

Consider spacetime with Minkowski metric $\eta_{\mu\nu}$. In Cartesian coordinates,

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

Now we replace the covariant derivatives by partial derivatives, and the Killing equation becomes

$$X_{a,b} + X_{b,a} = 0.$$

If we derive further, we get

$$X_{a,b\mu} + X_{b,a\mu} = 0.$$

Now, if we cycle the indices twice, we get

$$X_{b,\mu a} + X_{\mu,ba} = 0,$$
$$X_{\mu,ab} + X_{a,\mu b} = 0.$$

Next step is to add the first two and subtract the third. Then we have

$$0 = X_{a,b\mu} + X_{b,a\mu} + X_{b,\mu a} + X_{\mu,ba} - X_{\mu,ab} - X_{a,\mu b} = 2X_{b,a\mu}.$$

As seen above, the second derivative of X_b vanishes. This means that X_b must be linear in the coordinates,

$$X_a = \alpha_a + \beta_{ab}x^b.$$

If we substitute this into the Killing equation, we have

$$0 = X_{a,b} + X_{b,a} = \beta_{ab} + \beta_{ba},$$

where α_a is arbitrary while $\beta_{a\beta}$ is antisymmetric.

Therefore, we get 10 independent vector fields, each of the form

$$X_a = \alpha_a + \beta_{ab}x^b$$

for independent choices of the 10 constants α_a and $\beta_{ab} = -\beta_{ba}$.
If we take $\beta_{ab} = 0$ and one of the components (say, m for $m = 0, 1, 2, 3$) of α_a nonzero, we get four constant vector fields,

$$X^a_m = \delta^a_m.$$

This represents a unit vector in each of the coordinate directions. Since they are constant, the integral curves are just the Cartesian coordinate axes.

Now let set $\alpha_a = 0$ and choose one of the six antisymmetric matrices β_{ab}. Then we get either rotations or boosts. For instance, with $b_{21} = -b_{12} = 1$ and with all the rest zero, the vector field is

$$X = X^a\partial_a = (\eta^{ab}\beta_{b\mu}x^\mu)\partial_a = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

This is the generator of a rotation around the z axis. Analogously, $b_{23} = -b_{32}$ and $b_{31} = -b_{13}$ are the generators of rotations around the x and y axes, respectively.

Moreover, if one of the nonzero indices is time, then we get a boost because of the sign change. For $b_{10} = -b_{01} = 1$, we find

$$X = X^a \partial_a = (\eta^{ab} \beta_{b\mu} x^\mu) \partial_a = x \tfrac{\partial}{\partial t} + t \tfrac{\partial}{\partial x}.$$

This is a generator for a Lorentz transformation, which is a linear transformations from a coordinate frame in spacetime to another frame that moves at a constant velocity. To see this, exponentiate the generator with a parameter

$$\Lambda = \exp\left[\lambda\left(x \tfrac{\partial}{\partial t} + t \tfrac{\partial}{\partial x}\right)\right] = \textstyle\sum_{n=0}^{\infty} \tfrac{1}{n!} \lambda^n \left(x \tfrac{\partial}{\partial t} + t \tfrac{\partial}{\partial x}\right)^n.$$

Consider the effect on the coordinates (t, x, y, z). Clearly, $\Lambda y = \Lambda z = 0$. For t, we need

$$\left(x \tfrac{\partial}{\partial t} + t \tfrac{\partial}{\partial x}\right) t = x$$

$$\left(x \tfrac{\partial}{\partial t} + t \tfrac{\partial}{\partial x}\right)^2 t = \left(x \tfrac{\partial}{\partial t} + t \tfrac{\partial}{\partial x}\right) x = t$$

and so on, alternating between x and t. The even and odd parts of the series therefore sum separately

$$\Lambda t = \textstyle\sum_{n=0}^{\infty} \tfrac{1}{n!} \lambda^n \left(x \tfrac{\partial}{\partial t} + t \tfrac{\partial}{\partial x}\right)^n t = \textstyle\sum_{m=0}^{\infty} \tfrac{1}{(2m+1)!} \lambda^{(2m+1)} x + \tfrac{1}{(2m)!} \lambda^{(2m)} t = x \sinh\lambda + t \cosh\lambda.$$

Similarly, acting on x we get

$$\Lambda x = t \sinh\lambda + x \cosh\lambda.$$

We recognize λ as the rapidity, and the full transformation,

$$\begin{aligned} \Lambda t &= x \sinh\lambda + t \cosh\lambda \\ \Lambda x &= t \sinh\lambda + x \cosh\lambda \\ \Lambda y &= y \\ \Lambda z &= z \end{aligned}$$

as a boost in the x-direction.

Therefore, ten is the maximum number of independent solutions to the Killing equation, which means that we find exactly 10 symmetries in Minkowski space.

Chapter 4

Theory of perception

While our sensory receptors are constantly collecting information from the environment, it is ultimately how we interpret that information that affects how we interact with the world. Perception refers to the way sensory information is organized, interpreted, and consciously experienced. In this chapter, we first present possible models of perception. Furthermore, we introduce a mathematical model of perception. In last section, we prove Invention of Symmetry Theorem.

4.1 Models of perception

A relationship between perception and objective reality is called a perceptual strategy. Since before Plato, philosophers have proposed many theories of the relationship between perception and reality. Here are a few key theories.

4.1.1 Naïve realism

Perceptual scientists often claim that perceptions are accurate depictions of reality. S. E. Palmer [27], asserts that, "Evolutionary speaking, visual perception is useful only if it is reasonably accurate... By and large, what you see is what you get." Z. Pizlo and his collaborators [30] agree: "We close by re-stating the essence of our argument, namely, veridicality is an essential characteristic of perception and cognition. It is absolutely essential. Perception and cognition without veridicality would be like physics without the conservation laws." They also argue that creatures whose perceptions are more true, are also more fit. W. Geisler and R. Diehl [9] say, "In general, (perceptual) estimates that are nearer the truth have greater utility than those that are wide ofF the mark." Therefore, due to natural selection, the accuracy of perceptions, in most cases, approximate the truth. The model of perception that faithfully and exhaustively resembles reality is called naïve realism. It states that we see the truth, the whole truth and, most often, nothing but the truth.

4.1.2 Critical realism

The claim of naïve realism is weakensed by critical realism, also known as scientific realism. Critical realism states that perception faithfully resembles a part of reality, but not all of reality. We see the truth, but not the whole truth, and sometimes something other than the truth. For instance, we see visible light but not ultraviolet or x-rays, and we can have misperceptions, such as optical illusions. Most students of perception today are critical realists (J. T. Mark et al., [22]).

4.1.3 Interface theory of perception

D. D. Hoffman et al. [13] argue, on evolutionary grounds, that all of the above is false. They claim that instead, our perceptions constitute a species-specific user interface that guides behavior in an evolutionary niche. That model is called the Interface Theory of Perception (ITP) and it says:

Interface Theory of Perception: *The perceptions of an organism are a user interface between that organism and the objective world.*

ITP is illustrated intuitively using a graphical user interface (GUI) analogy. The desktop display of a computer shows a set of icons representing files, folders, operations (such as trash/delete), and apps. You don't take that interface literally. You understand that your latest manuscript isn't literally a little rectangle sitting in a clutter of other little rectangles in the upper left corner of your display. Rather that little rectangle is just a convenient icon that represents your manuscript. Similarly, you understand that if you drag that little rectangle to the trashcan in frustration, the rectangle isn't literally in the trashcan. Rather, the desktop is a convenient medium to interface with the underlying reality of your computer. It's useful because it hides the truth and instead presents a set of user-friendly shortcuts for writing papers, sending messages, and manipulating photos. Notice too how notions of causality play out in the interface. The cursor, the little rectangle, and the trashcan icons themselves have no causal power. It's not the movement of the target rectangle of your drag icon to the trashcan icon that causes the file to disappear; it's the underlying electric currents and switches that actually have causal power. According to ITP, the real causal powers are hidden from us. Moreover, an interface promotes efficient interaction with the computer by hiding its structural and causal complexity, i. e. by hiding the truth. Perception, Hoffmann et al. argue, is precisely the same: what we experience is nothing more than a set of species-specific icons, user-friendly shortcuts for staying alive and reproducing. To sum up, perception hides the truth and guides adaptive behaviors.

4.2 Mathematical model of perception

We now make mentioned theories of perception precise, beginning with a formal structure for perceptions. General reference for this section is J. T. Mark, B. B. Marion, D. D. Hoffman [22].

We assume that the external world can be represented by a measurable space, W. Similarly, we assume that a collection of perceptions can be represented as a set, X. As W, X is also a measurable space. A perceptual strategy is a map, or more precisely, a Markov kernel $P : W \to X$, that maps a state of the world to a random distribution of possible perceptions of that state by an agent. Different perceptual strategies differ in the properties of P.

For the simplest version of a naïve realist strategy, $X = W$ and P is an isomorphism. Perception faithfully resembles all of reality. Furthermore, we distinguish two types of critical realist strategies: strong and weak. Strategies of the strong type are a proper subset of strategies of the weak type. For the strong type, $X \subset W$ and P is an isomorphism. Perception faithfully mirrors a subset of reality. For the weak type, $P : R \subset W \to P(R)$ is a homomorphism. Perception need not faithfully mirror any subset of reality, but relationships among perceptions reflect relationships among aspects of reality. Thus, weak critical realists can be based on utility, so long as this homomorphism is maintained. For the interface strategy, in general P need not be an isomorphism or even a homomorphism on any subset of W. Perception need not faithfully mirror any subset of reality, and the relationships among perceptions need not reflect relationships among aspects of reality.

Given these definitions, the naïve realist strategies are a subset of the critical realist strategies, which in turn are a subset of the interface strategies, as illustrated in Figure 3.1.

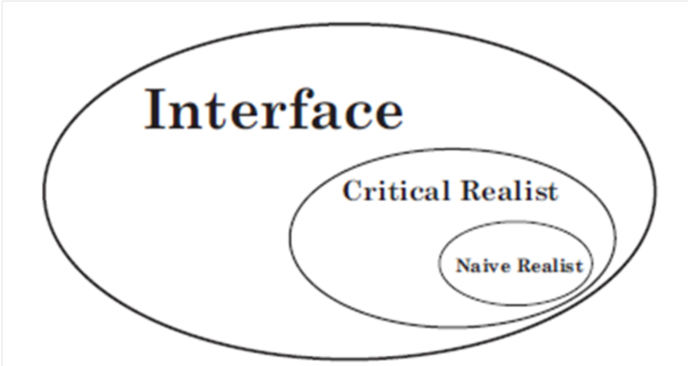


Figure 4.1: Classes of perceptual strategies. [Source: J. T. Mark, B. B. Marion, D. D. Hoffman, [22]]

We wish to study the fitness of these three classes of perceptual strategies in an evolutionary environment. For this, we turn to evolutionary games. In Chapters 5, 6 and 7 we present mathematical models of above mentioned perceptual strategies and study the competition between them. But before that, in the next section, we prove Invention of Symmetry Theorem by D. D. Hoffman, M. Singh, C. Prakash [14], that integrates the content of Chapters 3 and 4 into a mathematical model of a significant philosophical question: 'What is reality?'.

4.3 Invention of Symmetry Theorem

We can expand our perceptions of spacetime by using groups symmetry, e. g., Euclidean groups. Changes in an observer's viewpoint can then be modeled by actions of these groups on the appropriately extended spacetime. General reference for this section is D. D. Hoffman, M. Singh, C. Prakash [14].

B. Russell [34] claims that if a feature of our perceptions is invariant under these group actions, then it can be taken as veridical. But this claim is false. The Invention of Symmetry Theorem shows that the world itself might not share any of the observed symmetries. The world need not have the structure percieved by the observer, no matter how complex that structure is and no matter how predictably and systematically that structure transforms.

Theorem 4.1. *(Invention of Symmetry Theorem) Let an observer have at its disposal a group G of actions on the world W, such that observer's perceptual space X is a G-set. This means that G acts on X via the kernel $PA = P(A(g)) = \int P(w, dx)A(g, dw)$, i. e. the action of A followed by that of P; moreover G acts on X by a transitive group action, so that G is a symmetry group of X. Let G act on W in such a way that the observer's perceptual channel mediates this action: $P(g.w) = g.P(w)$, where the dot signifies the action of G on* *each set. Then, the perceptual experiences X of this observer will admit a structure with G as its group of symmetries.*

Definition 4.2. *The points, $w, w' \in W$, are in the same fiber if the probability measure $P(w, \cdot)$ on $(X, \mathcal{X})$ is the same as the probability measure $P(w', \cdot)$. In other words, w and w' are indistinguishable to the observer.*

Proof. Let S_x be the fiber of P over $x \in X$. Then, we may view $W = \cup_{x \in X} S_x$ and think of each element of W as a pair (x, s), $s \in S_x$. Because the function P is onto X, we can view P as a projection: $P(x, s) = x$. When G acts on W, it will take each element (x, s), $s \in S_x$, to an element $(g.x, s')$, $s' \in S_{g.x}$. This preserves the fibers of P. Moreover, when G acts on W via the group element g, because of $g.x = g.P(w) = P(g.w)$, it automatically acts on X by the same element. □

Perceptual experiences of an observer may have a rich structure, e. g., a 3D structure that is locally Euclidean, and that transforms predictably and systematically as the observer acts, but this says absolutely nothing about the structure of the world. This is entirely contradictory to instinct. We naturally assume that the rich structure of our perception, and their transformations as we act, must be an insight into the true structure of the objective world. Invention of Symmetry Theorem shows that our intuition here is completely wrong.

Chapter 5

Interface perceptual strategies drive veridical strategies to extinction

J. T. Mark et al. [22] use evolutionary games to analyze under what circumstances natural selection favors veridical perceptions. The goal of evolutionary game theory is to explain behavior in strategic settings typically biological or economic from the perspective of Darwinian evolution. An evolutionary system involves three basic elements: a strategic game, a population of players, and some mathematical conception of the evolution of strategic game-play throughout the population. We reproduce mathematical model for territorial games introduced by J. T. Mark et al. [22] and adapt it to new perceptual strategies.

5.1 Mathematical model

In a territorial game, p players compete in pairs, they choose between t territories with k resources each. Each resource takes a discrete value in the set $V = \{1, 2, ..., m\}$. Let r_T be the vector of resources in territory T. The utility of territory T is defined as

$$u(T) = U(r_T) = \sum_{i=1}^{k} U_i(r_{T,i}), \tag{5.1}$$

where U_i is the utility of contribution of resource $r_{T,i}$ in territory T. In our investigations, it is either monotonous linear or Gaussian, as we specify later.

From interface theory of perception combined with utility theory, we take the universal model of an agent – a process, and from game theory, we take the decision tree to be the structure of decisions of each agent. Figure 5.1 shows a universal process model, which creates an image of the world using perception methods (measurement, observation, communication). On this basis, decisions between possible activities are made. The implementation of these activities leads to a change in the state of the world, which completes the perception-decision-implementation cycle. Within the state of the world, we can evaluate the usefulness of the state of the world or utility of system operation. (D. Bokal, P. Fic [4]) The agent acting in the world W is defined with a 5-tuple $A = (X, G, P, D, U)$, where

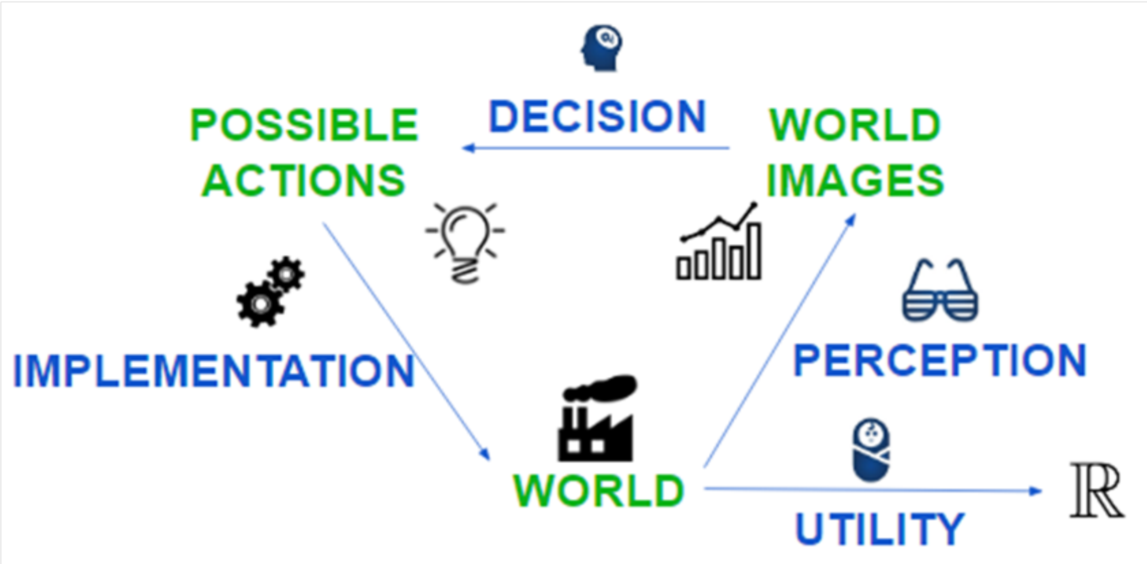


Figure 5.1: Universal model of a process. [Source: (D. Bokal, P. Fic [4]).]

X is the space of agent's possible perceptions, G is a semigroup of agent's actions. Similarly as W, both X and G are measurable spaces. Then P, D, U are agent's perception, decision, and utility operators. In highest generality, they are modelled as Markov kernels; $P : W \to X$, $D : X \to G$, $U : W \to \mathbb{R}$. Actions $A \in G$ are defined as $A : W \to W$, and they are Markov kernels as well. This model fitting the universal model of a process is shown on Figure 5.2.

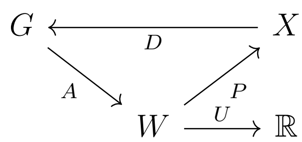


Figure 5.2: Agent acting in the world. [Source: Own.]

For the rest of this section, we assume that $r = 1$. J. T. Mark et al. [22] introduce four agent strategies that could participate in one of two games that differ in the definition of utility. Utility is mapping $U : W \to \mathbb{R}$ defined as either identity in the first game or Gaussian function in the second game:

$$U(x) = \frac{1}{\sqrt{40\pi}} e^{-\frac{1}{2}\frac{(x-50)^2}{400}} \tag{5.2}$$

Moreover, for all four strategies described next, world W is the same and it is defined as $W = \{1, \ldots, m\}^t$.

As described in Chapter 3, a naïve realist perception faithfully and exhaustively resembles reality. The mapping from the world to the space of perceptions ($X_n = W$) of naïve realist is an identity map: $P_n : W \to X_n$. Decision is a mapping from space of perceptions to a semigroup of agent's actions $G_n = \{1, \ldots, t\}$ $D_n : X \to G_n$ implying that the agent picks the available territory with maximum amount of the resource, and defined as:

$$D_n(X) = \operatorname{argmax}_{p=P_n(w), i \in G_n} p[i]. \tag{5.3}$$

The mapping from the world to the world, $A : W \to W$, named actions is for a naïve realist defined as the change of utility and the change of a resource value on the territory:

$$\begin{aligned} u' &= u + U(r_i), \\ r_i' &= 0. \end{aligned} \tag{5.4}$$

A mathematical model for a naïve realist is shown in Table 5.1:

Elt	**naïve realist**
W	$\{1, \ldots, m\}^t$
X	W
G	$\{1, \ldots, t\}$
P	w
A	$u' = u + U(r_i), r_i' = 0$
D	$\operatorname{argmax}_{p=P_n(w), i \in G_n} p[i]$
U	$U(r_i)$

Table 5.1: Mathematical model of a naïve realist.

Next we introduce a mathematical model of a semi-veridical strategy, the critical realist CRn. He does not perceive the true amount of resource, but categorizes it into n categories preserving the ordering. J. T. Mark et al. [22] call this a $nCat$ agent. Assume that $n = 2$. Critical realist claims that perception faithfully resembles some aspect of reality, but not all of it: he distinguishes between little and much of the resource. The mapping from the world to its space of perceptions is $P_c : W \to X_c$, $X_c = \{0, 1\}$, defined as:

$$P_c(w) = \begin{cases} 0, & \text{if } 1 \leq w \leq \beta, \\ 1, & \text{if } \beta < w \leq m, \end{cases} \tag{5.5}$$

where β is a boundary between perceiving lots or little of the resource. Decision is a mapping from the space of perceptions to a semigroup of agent's actions $D_c : X_c \to G_c$, $G_c = X_c$ given by:

$$D_c(X) = \text{argmax}_{p=P_c(w), i \in G_c} \, p[i]. \tag{5.6}$$

Among maximal elements with $p[i] = p[j]$, the territory is chosen randomly and holds also for all subsequently described strategies. Actions, the mapping from the world to the world, $A : W \to W$, are for $CR2$ defined as:

$$\begin{aligned} u' &= u + U(r_i), \\ r'_i &= 0. \end{aligned} \tag{5.7}$$

Now assume that $n = 3$. A critical realist with three perceptual categories (here highlited with red, yellow, green) and perceptual order red < yellow < green is illustrated in Figure 5.3; J. T. Mark et al. [22] calls it CR3. Its decision rule would be to prefer yellow to green, and green to red. Notice that the order governing its decision rule is red < green < yellow which differs from its perceptual order.

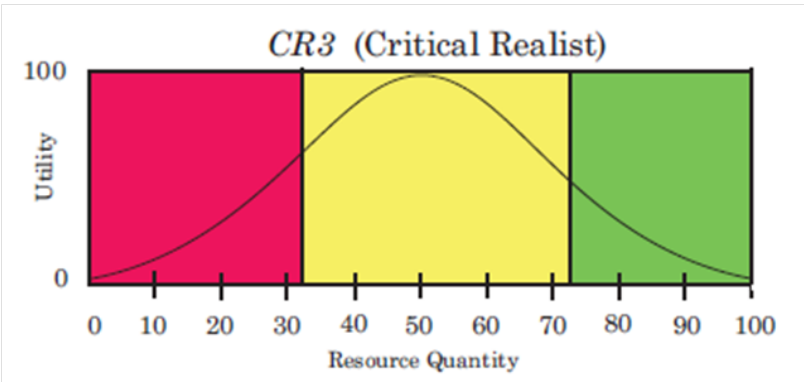


Figure 5.3: Optimal boundary placement on a Gaussian utility structure for a $3Cat$ critical realist. [Source: (J. T. Mark, B. B. Marion, D. D. Hoffman, [22]).]

A mathematical model of a $3Cat$ critical realist strategy maps from the world to its space of perceptions with $P_p : W \to X_p$, $X_p = \{0, 1, 2\}$, defined as:

$$P_p(w) = \begin{cases} 0; & \text{if } 1 \le w \le \beta_1, \\ 1; & \text{if } \beta_1 < w \le \beta_2, \\ 2; & \text{if } \beta_2 < w \le m, \end{cases} \tag{5.8}$$

where β_1 and β_2 are boundaries in CR3's perception between low, intermediate, and high amounts of the resource. Decision is a mapping from space of perceptions to a semigroup of agent's actions $D_p : X_p \to G_p$ given by priority partial order: $0 < 2 < 1$, i. e. it chooses randomly between territories whose perceived amounts of resource are maximal elements for this partial order. Actions for CR3 are the same as for the CR2 (5.7).

In general, we define a mathematical model of a $nCat$ critical realist strategy (CRn) as:

Elt	**CRn**
W	$\{1, \ldots, m\}^t$
X	$\{0, \ldots, n-1\}^t$
G	$\{1, \ldots, t\}$
P	$\begin{cases} 0; & \text{if } 1 \le w \le \beta_1, \\ 1; & \text{if } \beta_1 < w \le \beta_2, \\ ... & \\ n-1; & \text{if } \beta_{n-1} < w \le m, \end{cases}$
A	$u' = u + U(r_i), r'_i = 0$
D	$0 < n-1 < ... < 1$
U	$U(r_i)$

Table 5.2: Mathematical model of a CRn.

A $3Cat$ utilitaristic interface perception that we introduce next is used by Hoffman et al. [14] to argue that perceptions need not, and in general do not, resemble any aspect of reality. In Figure 5.4, there is an interface strategy illustrated. It has four boundaries, but only three perceptual categories (here highlited with red, yellow, green), so we call it IF3. It's decision order is red < yellow < green.

We introduce a mathematical model of IF3. The mapping from the world to its space of perceptions is a map: $P_i : W \to X_i$, $X_i = \{0, 1, 2\}$ defined as:

$$P_i(w) = \begin{cases} 0; & \text{if } 1 \le U(w) \le \beta_1, \\ 1; & \text{if } \beta_1 < U(w) \le \beta_2, \\ 2; & \text{if } \beta_2 < U(w) \le m, \end{cases} \tag{5.9}$$

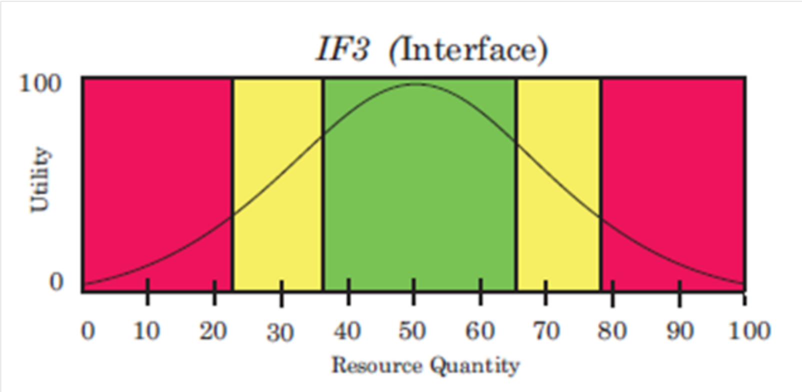


Figure 5.4: Optimal boundary placement on a Gaussian utility structure for a $3Cat$ interface strategy. [Source: (J. T. Mark, B. B. Marion, D. D. Hoffman, [22]).]

where β_1 and β_2 are boundaries in IF3's perception between insufficient, mediocre, and very useful amount of resource. Decision is a mapping from space of perceptions to a semigroup of agent's actions $D_i : X \to G$ given by:

$$D_i(X) = \underset{p=P_i(w), i \in G_i}{argmax} \; p[i]. \tag{5.10}$$

Actions, the mapping from the world to the world, $A : W \to W$, are for IF3 defined as:

$$\begin{aligned} u' &= u + U(r_i), \\ r_i' &= 0. \end{aligned} \tag{5.11}$$

In general, we can define a mathematical model for a $nCat$ interface strategy. The details of this model, reproduced from J. T. Mark et al. [22], are given in Table 5.3:

Elt	**IFn**
W	$\{1, \ldots, m\}^t$
X	$\{0, \ldots, n-1\}^t$
G	$\{1, \ldots, t\}$
P	$\begin{cases} 0; & \text{if } 1 \leq U(w) \leq \beta_1, \\ 1; & \text{if } \beta_1 < U(w) \leq \beta_2, \\ \ldots & \\ n-1; & \text{if } \beta_{n-1} < U(w) \leq m, \end{cases}$
A	$u' = u + U(r_i), r_i' = 0$
D	$argmax_{p=P_i(w), i \in G_i} \, p[i]$
U	$U(r_i)$

Table 5.3: Mathematical model of IFn.

5.2 Evolutionary dynamics

Evolutionary game theory can be used to study long term interactions between two strategies. For instance, when does one strategy drive the other to extinction, and when do the two stably coexist? To this end, we create a payoff matrix describing the competition between the strategies, as shown in Table 5.4:

	A	**B**
A	a	b
B	c	d

Table 5.4: Payoff matrix

The payoff matrix is defined as follows. Agent A gets payoff a when competing against agent A and payoff b when competing against agent B. Agent B gets payoff c when competing against agent A and payoff d when competing against agent B. The payoff to a strategy is taken to be the fitness of that strategy, i. e. its reproductive success.

In order to determine the long-term outcome of competition between two agent strategies (A) and (B), we apply the following proposition.

Proposition 5.1. *(J. J. Armao, [1]) Let a be the expected utility obtained by (A) competing with (A), b the expected utility of (A) competing with (B), c the expected utility of (B) competing with (A) and d the expected utility of (B) competing with (B). There are four possible long term outcomes of the evolutionary competition of the two species.*

1. *If $a \geq c$ and $b \geq d$ and at least one inequality is strict, then (A) prevails and (B) goes extinct.*
2. *If $c \geq a$ and $d \geq b$ and at least one inequality is strict, then (B) prevails and (A) perishes.*
3. *If $a < c$ and $b > d$, they stably coexist.*
4. *If $a > c$ and $b < d$, they are bistable, i. e. each is asymptotically stable and which depends on the initial conditions.*
5. *They are neutral, if $a = c$ and $b = d$, i. e. their prevalence changes randomly.*

A strategy "wins" if it drives the other strategy to extinction regardless of the initial proportions of the strategies; a winning strategy is the best response to itself and the other strategy. Two strategies stably coexist if, independent of their initial proportions, those proportions approach asymptotically stable values; each strategy is the best reply to the other strategy, but not to itself. Two strategies are bistable if their initial proportions determine which strategy drives the other to extinction; each strategy is the best response to itself, but not to the other strategy. Two strategies are neutral if their initial proportions, whatever they happen to be, are preserved asymptotically (J. T. Mark, B. B. Marion, D. D. Hoffman, [22]).

The following Lemma implicitly applied in J. T. Mark et al., [22] tells us how to determine expected payoffs.

Lemma 5.2. *Let the interaction of (A) and (B) be as described, and let p be the probability that (A) moves first when interacting with (B). Furthermore, let* $\Gamma(i, X, Y)$ *denote the expected utility of the i-th player, where* $X \in \{A, B\}$ *is the strategy that chooses first, and* $Y \in \{A, B\}$ *is the strategy that chooses second. Then,*

1. $a = \frac{1}{2}\left(E(\Gamma(1, A, A)) + E(\Gamma(2, A, A))\right)$,
2. $b = pE(\Gamma(1, A, B)) + (1 - p)E(\Gamma(2, B, A))$,
3. $c = pE(\Gamma(2, A, B)) + (1 - p)E(\Gamma(1, B, A))$, *and*
4. $d = \frac{1}{2}\left(E(\Gamma(1, B, B)) + E(\Gamma(2, B, B))\right)$.

5.3 Cost of information

Every strategy uses a specific amount of energy to percieve and handle data. Moreover, different strategies use different amounts of energy since processing more information takes, on average, more time and energy. The energy expenditure that each strategy uses to acquire and process information, and to choose a territory, is called cost of information. This cost is computed by multiplying the cost per bit of information, c_e, by the number of bits used. There are also, of course, energy costs for decision, not just for perception. But for simplicity of analysis, we ignore these here. (J. T. Mark, B. B. Marion, D. D. Hoffman, [22])

Naïve realist sees $\log_2(m)$ bits of information per territory, which for $m = 100$ is approximately 20 bits. Therefore, because there are 3 territories, it sees a total of $3 \log_2(m)$ bits of information. Cost of information for a naïve realist is then $3c_e \log_2(m)$.

Critical $2Cat$ realist sees 1 bit of information per territory, since it only distinguishes between little and much of the resource. Therefore, because there are 3 territories, it sees a total of 3 bits of information. Cost of information for a critical realist is then $3c_e$.

$nCat$ agents receive the highest payout by choosing territories with resources falling inside one of the perceived ranges, rather than by finding the greatest quantity of resources.

In order to pick the best territory, these strategies must know the payout associated with each possible perceived resource quantity. Therefore, we must now charge each strategy for both seeing the quantity and knowing the utility of each resource, and its cost becomes

$$c_e[trlog_2(q)] + c_k[rqn_b], \tag{5.12}$$

where c_e is again the cost per bit of information, c_k is the cost per bit of knowledge about utility values, t is the number of territories, r is number of resources, q is the number of perceptual categories for that strategy, and n_b is the number of bits used to represent the utility of a resource quantity. Since CR3 and IF3 only need to order their three categories, their n_b in (5.12) is $\log_2(3)$. We also assume that $c_k = \frac{c_e}{10}$, making energy costs of perception and knowledge of utility roughly on a par.

5.4 Results: Veridical strategies driven to extinction

J. T. Mark et al. [22] first explore a competition between naïve realist and critical realist CR2.

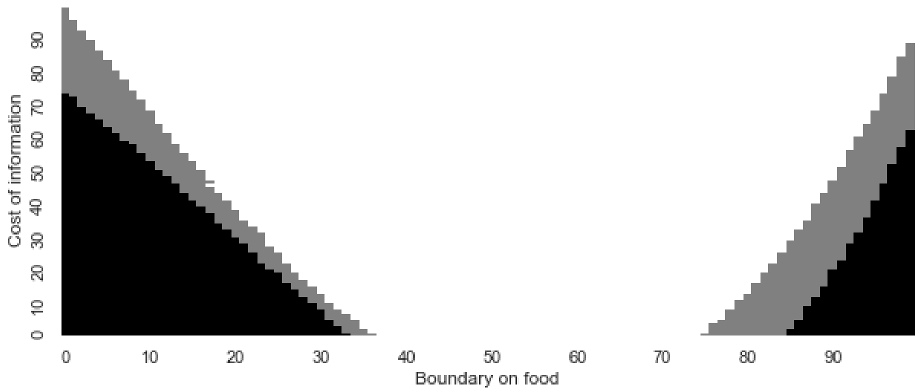


Figure 5.5: The results of an evolutionary game between naïve realist and critical realist with a single resource. [Source: Reproduced from [22]]

Figure 5.5 shows how the cost of information and a threshold on food used by a $2Cat$ critical realist in perceiving the world affects the competition of $2Cat$ critical realist and naïve realist. A critical realist drives a naïve realist to extinction (white color) for most of the values of threshold β. Only for low cost of information, naïve realist drives critical realist to extinction (black color), and only when the threshold β is chosen poorly. On gray area, the two strategies stably coexist.

J. T. Mark et al. [22] also explore a 3-way competition between naïve realist, CR3 and IF3 strategies in the setting of non-monotonous Gaussian utility. They show that either veridical or IF3 strategy outperform the other two, depending on the cost of information. We reproduce their results only competing CR3 with IF3, but varying the width of the central interval that CR3 perceives as intermediate and imposing some cost on perceiving the true utility. On Figure 5.6 we can see the results: the augmented critical realist has a slight

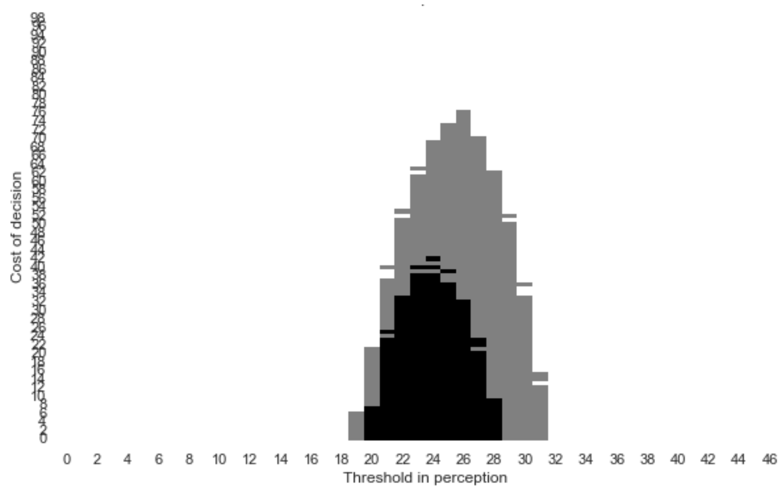


Figure 5.6: The results of an evolutionary game between CR3 and IF3 with a single resource. Note that y-scale is 1:500 compared to Figure 5.5. [Source: Own.]

chance with narrow enough intermediate interval, (the scale of the y axis is 1:500 compared to Figure 5.5). The interface strategy drives the critical realist to extinction (white color) for most of the interval width. Only for narrow interval and very low cost of information, critical realist drives interface strategy to extinction (black color), although adjusting the parameters of the IF3 perceptive strategy may possibly compensate for that advantage.

In summary, these competitions show that natural selection does not always favor naïve realism or critical realism, and that in many scenarios, only the interface strategy survives.

Chapter 6

Innovative veridical perceptual strategies drive interface strategies to extinction

In addition to the experiments using Gaussian utility described in Chapter 5, we define two new instances of agents that have the possibility of storing surplus resources, i. e. effectively they introduced a new dimension into the world. The basis for the introduction of such strategies is evolutionary: an IF3 mutation, which otherwise perceives the quantity of the resource in a utilitarian manner, can develop the possibility of storing excessive amounts of the resource. The agent using such strategy needs to decide, whether to consume the resource or store it. For such a decision, the agent needs to distinguish between too much and too little of the resource, and needs a credible perception of its amount: it is evolutionarily motivated for veridical perception.

Strategies with storage can be divided according to how the reality is perceived into critical realist strategy with n categories (CRn) and interface strategies with n categories (IFn). Analogously, storage can be perceived in two ways: critically realistic with m categories (SRm) or utilitaristically with m categories (SIFm).

In our model, the amount of stored resource (r_s) takes value in the set $V = \{1, 2, ..., m\}$. $3Cat$ critical realist with storage perceives the amount of resource in the storage as $2Cat$ critical realist would perceive another territory, i. e. there is a threshold value r^*, such that the storage is seemed empty for $r_s < r^*$, or full otherwise. In each case, the CR3SR2 agent chooses the territory with intermediate amounts of the resource first, so as to take advantage of the inherent utility of the territory first. If there is none and the storage is empty, the agent prefers the territory with too much of the resource; if the storage is perceived full, the agent prefers territory with too little of the resource.

For the storage to be fully exploited, we need to introduce another parameter into the model:

Definition 6.1. *Lifespan l is defined as the number of interactions between the two competing strategies.*

Lifespan says how often can two agents compete for a territory and apply the advantage of storing excessive or supplementing insufficient amount of the resource.

6.1 Mathematical model of a CR3SR2

We now introduce our mathematical model of a $3Cat$ critical realist with storage (CR3SR2).

A mathematical model of a $3Cat$ critical realist strategy with storage (CR3SR2) is defined on the measure space $W = \{1, \dots, m\}^{t+1}$. It maps from the world to its space of perceptions with $P_{cs} : W \to X_{cs}$, $X_{cs} = \{0, \dots, 2\}^t \times \{0, 1\}$, defined as:

$$P_{cs} = (P_c, P_s); P_c(t) = \begin{cases} 0; & \text{if } 1 \leq t \leq \beta_c, \\ 1; & \text{if } \beta_c < t \leq \gamma_c, \\ 2; & \text{if } \gamma_c < t \leq m. \end{cases} P_s(s) = \begin{cases} 0; & \text{if } 1 \leq s \leq \beta_s, \\ 1; & \text{if } \beta_s < s \leq m, \end{cases} \tag{6.1}$$

where β_c and γ_c are boundaries in CR3SR2's perception between low, intermediate, and high amounts of the resource and β_s is boundary in CR3SR2's perception between empty and full storage.

Decision is a mapping from space of perceptions to a semigroup of agent's actions $D_{cs} : X_{cs} \to G_{cs}$ given by priority partial order:

$$D_{cs} = \begin{cases} 0 < 1 < 2; & \text{if } 1 \leq s \leq \beta_s, \\ 0 < 2 < 1; & \text{if } \beta_s < s \leq m, \end{cases} \tag{6.2}$$

which depends on the state of storage.

Actions, the mapping from the world to the world, $A_{cs} : W \to W$, are for CR3SR2 given by change of the amount of resource in storage and utility:

$$s' = \begin{cases} \min(s_{max}, r_i + s - r^*); & \text{if } r_i + s > r^*, \\ \max(0, r_i + s - r^*); & \text{if } r_i + s \leq r^*, \end{cases} \tag{6.3}$$

$$u' = u + \begin{cases} U(r^*); & \text{if } r_i + s > r^*, \\ U(r_i + s); & \text{if } r_i + s \leq r^*, \end{cases}, r_i' = 0. \tag{6.4}$$

Utility U, which is a mapping from the world to the set of a real numbers, is defined as $U(r_i)$.
The details of mathematical model of a CR3SR2 are given in Table 6.1.

Elt	**CR3SR2**
W	$\{1, \ldots, m\}^{t+1}$
X	$\{0, \ldots, 2\}^t \times \{0, 1\}$
G	$\{1, \ldots, t\}$
P	$P_c(t) = \begin{cases} 0; & \text{if } 1 \leq t \leq \beta_c, \\ 1; & \text{if } \beta_c < t \leq \gamma_c, \\ 2; & \text{if } \gamma_c < t \leq m. \end{cases}$ $P_s(s) = \begin{cases} 0; & \text{if } 1 \leq s \leq \beta_s, \\ 1; & \text{if } \beta_s < s \leq m, \end{cases}$
A	$s' = \begin{cases} \min(s_{max}, r_i + s - r^*); & \text{if } r_i + s > r^*, \\ \max(0, r_i + s - r^*); & \text{if } r_i + s \leq r^*, \end{cases}$
	$u' = u + \begin{cases} U(r^*); & \text{if } r_i + s > r^*, \\ U(r_i + s); & \text{if } r_i + s \leq r^*, \end{cases}, r_i' = 0$
D	$\begin{cases} 0 < 1 < 2; & \text{if } 1 \leq s \leq \beta_s, \\ 0 < 2 < 1; & \text{if } \beta_s < s \leq m, \end{cases}$
U	$U(r_i)$

Table 6.1: Mathematical model of CR3SR2.

6.2 Cost of information

Next to the earlier defined lifespan, there is another new parameter introduced by the game. It is the cost that having a storage incurs on the individual. Since CR3SR2 has a storage, and IF3 does not have a storage, CR3SR2 uses more energy. Therefore, we must charge CR3SR2 for seeing the quantity in storage, and its cost becomes

$$c_e[c_0 q], \tag{6.5}$$

where c_e is again the cost per bit of information, c_0 is the cost per bit of knowledge about utility values, q is the number of perceptual categories of storage for that strategy. We somewhat arbitrarily assume that $c_0 = \frac{M_v(50)}{800}$.

6.3 Results: IF3 perishes against CR3SR2

While the results of J. T. Mark et al. [22] and Invention of Symmetry Theorem may shed doubt into veracity of our understanding of the world, we present further similar experiments that exhibit a significantly more complex structure of the problem of veracity of perceptions: we claim the dynamics of the evolution may create circumstances under which the increasing veracity of perceptions may be favourable in comparison to utilitaristic perceptions. Suppose a mutation in the IF3 organisms would introduce storage of the excessive amount of the resource that the organism cannot utilitarize. Then, it would be favourable for the organism to distinguish between too much and too little of the resource in the territory, thus being able to utilitarize the storage better.

In this section, we show that CR3SR2 can successfully compete with interface strategy IF3. Organism applying this innovation would move more slowly, as the storage would handicap its agility. We explore mentioned competition in different lifetimes, where lifetime is a number of interactions. Specifically, lifetime of lenght 1 means that the storage is empty at first and that agents compete only once. Analogously, the lifetime of the length l means that the storage is empty at first and it participates in every new competition with storage from the previous interaction.

We competed IF3 against CR3SR2 for lifetimes $l \in \{1, 2, \ldots, 32, 64, 128, 250, 500, 1000\}$, at which value the results converged and the last two images showed no further change. Each was competing at discrete values of $r_s^* \in \{1, 11, 21, 31, \ldots, 91, 101\}$, with the last value interpreted as always perceiving empty storage and desiring to claim territory with excessive amounts of the resource. For each of these values, we imposed a cost on the storage of $CR3SR2$ competing with $IF3$ with values $c_s \in \{1, 11, 21, 31, \ldots, 91, 101\}$, which was comparable to the maximum amount of utility U_v, taking values in the interval $[0.0039, 0.0892]$.

Figure 6.1 shows the results for given values of r_s^* (x axis) and c_s (y axis) for lifespan of $l = 4$ interactions, where the benefit of the storage is the highest. We see that CR3SR2 drives IF3 to extinction (black color) for all examined low values of storage cost and coexist for all the higher. Although CR3SR2 is slower at picking the territory, storage gives it the advantage over the utilitaristic IF3 when cost of storage is comparable to the utility of a

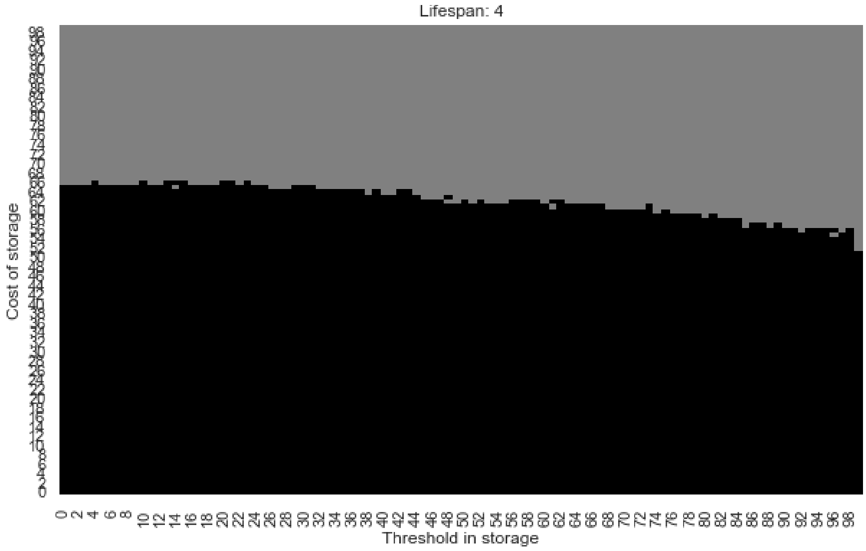


Figure 6.1: The results of an evolutionary game between CR3SR2 and IF3 with a single resource for lifespan 4 years.

terrain.

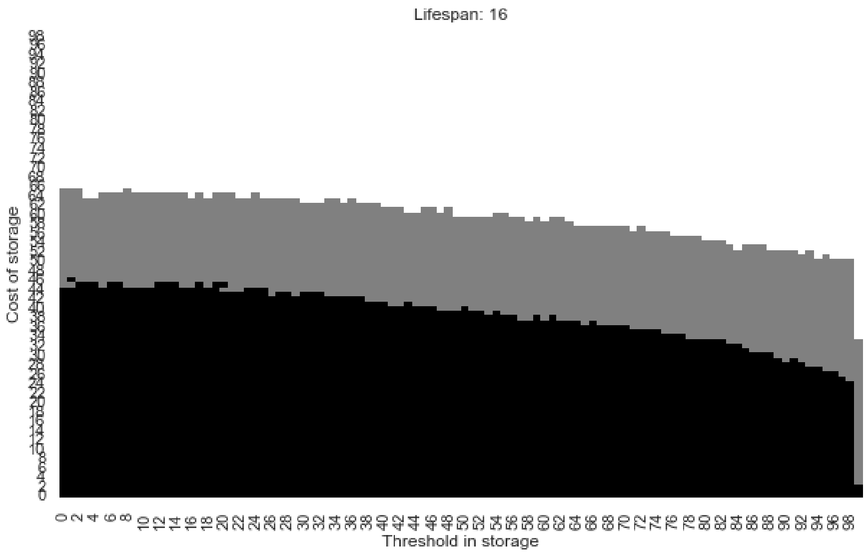


Figure 6.2: The results of an evolutionary game between CR3SR2 and IF3 with a single resource for lifespan 16 years. [Source: Own.]

As shown on Figure 6.2, for higher lifespans, the advantage of the storage decays. Furthermore, it converges with $l = 500$, but for lower values of c_s, CR3SR2 still dominates IF3 for all values of r_s^*.

To sum up, we showed that there exist conditions in which (simplified) veridical perceptions can drive utilitaristic strategies to extinction.

Chapter 7

Innovative interface strategies drive innovative veridical strategies to extinction

We introduce another mutation in the winning veridical perception and allow it to perceive the exact utility of the terrain and the current storage combined in a similar way the IF3 strategy perceives the utility of just the terrain.

In addition to CR3SR2, we introduce a 3Cat interface strategy with storage, IF3S. This strategy has storage, but perceives the exact utility of the territory given the exact amount of resource available in the storage. The utility is perceived in three categories as with IF3, and the decision operator is the same. We hence term this to be an interface strategy, although it needs veridical inputs to produce utilitaristic perception: both territory amount and storage amount of the resource must be accounted for veristically to distinguish between the three categories of utility in $3Cat$ perceptions.

7.1 Mathematical model of IF3S

We will now introduce our mathematical model of a $3Cat$ interface strategy with storage (IF3S), who perceives the exact utility received by using the storage innovation in the same way as IF3, i. e. it computes the utility $u(r, s)$ received when the selected territory has r amount of resource and s is the amount in the storage, and then perceives that amount of utility using the perceptions of IF3, i. e. very useful, mediocre, insufficient.

A mathematical model of a $3Cat$ interface strategy with storage (IF3S2) is defined on the measure space $W = \{1, \ldots, m\}^{t+1}$. It maps from the world to its space of perceptions with $P_u : W \to X_u$, $X_u = \{0, \ldots, 2\}^t \times \{0, 1\}$, and is defined as:

$$P_u(t, s) = \begin{cases} 0; & \text{if } 0 \leq U_c(t, s) \leq \beta_u, \\ 1; & \text{if } \beta_u < U_c(t, s) \leq \gamma_u, \\ 2; & \text{if } \gamma_u < U_c(t, s). \end{cases} \tag{7.1}$$

where β_u and γ_u are boundaries in IF3S's perception between insufficient, mediocre, and very useful amount of utility in regards to the state of storage.

Decision is a mapping from space of perceptions to a semigroup of agent's actions $D_u : X_u \to G_u$, $G_u = \{1, \ldots, t\}$ given by priority partial order:

$$0 < 1 < 2. \tag{7.2}$$

Actions, the mapping from the world to the world, $A_{cs} : W \to W$, are for CR3SR2 given by change of the amount of resource in storage and utility:

$$s' = \begin{cases} \min(s_{max}, r_i + s - r^*); & \text{if } r_i + s > r^*, \\ \max(0, r_i + s - r^*); & \text{if } r_i + s \leq r^*, \end{cases} \tag{7.3}$$

$$u' = u + \begin{cases} U(r^*); & \text{if } r_i + s > r^*, \\ U(r_i + s); & \text{if } r_i + s \leq r^*, \end{cases}, r_i' = 0 \tag{7.4}$$

Utility U, which is a mapping from the world to the set of a real numbers is defined as $U(r_i)$.

Details of our mathematical model of a $3Cat$ interface strategy with storage (IF3S) are given in Table 7.1.

Elt	IF3S
W	$\{1,\ldots,m\}^{t+1}$
X	$\{0,\ldots,2\}^t \times \{0,1\}$
G	$\{1,\ldots,t\}$
P	$P_u(t,s) = \begin{cases} 0; & \text{if } 0 \leq U_c(t,s) \leq \beta_u, \\ 1; & \text{if } \beta_u < U_c(t,s) \leq \gamma_u, \\ 2; & \text{if } \gamma_u < U_c(t,s). \end{cases}$
A	$s' = \begin{cases} \min(s_{max}, r_i + s - r^*); & \text{if } r_i + s > r^*, \\ \max(0, r_i + s - r^*); & \text{if } r_i + s \leq r^*, \end{cases}$
	$u' = u + \begin{cases} U(r^*); & \text{if } r_i + s > r^*, \\ U(r_i + s); & \text{if } r_i + s \leq r^*, \end{cases}, r_i' = 0$
D	$0 < 1 < 2$
U	$U(r_i)$

Table 7.1: Mathematical model of IF3S.

7.2 Cost of information

When CR3SR2 competed with IF3S, we applied the same values for r_s^*, but imposed additional costs with the same values on IF3S. We assumed that IF3S would spend more energy for the additional computations in the perception mechanism, but the cost of storage would be the same for both strategies. Therefore, we must charge IF3S, and its cost becomes

$$\frac{U_v(50)}{1000} c_e, \tag{7.5}$$

where c_e is again the cost per bit of information.

7.3 Results: CR3SR2 perishes against IF3S

In this section, we demonstrate that interface perception with storage has significant advantage over CR3SR2, thus innovative utilitaristic interface perception is the final evolutionary winner.

In Figure 7.1, we show the results of the competition between CR3SR2 and its mutation IF3S, which is able to precisely perceive the utility of combined amount of the resource in its storage and a possible territory. We assume this innovative storage imposes some additional cost to the organism. The figure shows success of the innovative utilitaristic interface perception over the original critical realist, which is emphasized with higher number

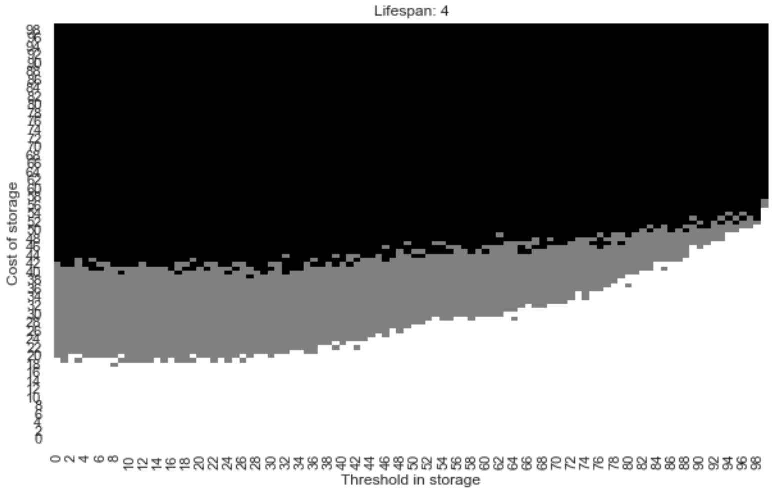


Figure 7.1: The results of an evolutionary game between CR3SR2 and IF3S2 with a single resource for lifespan 4 years. [Source: Own.]

of average interactions between the organisms: in comparison to rather comparable success of both strategies at four interactions.

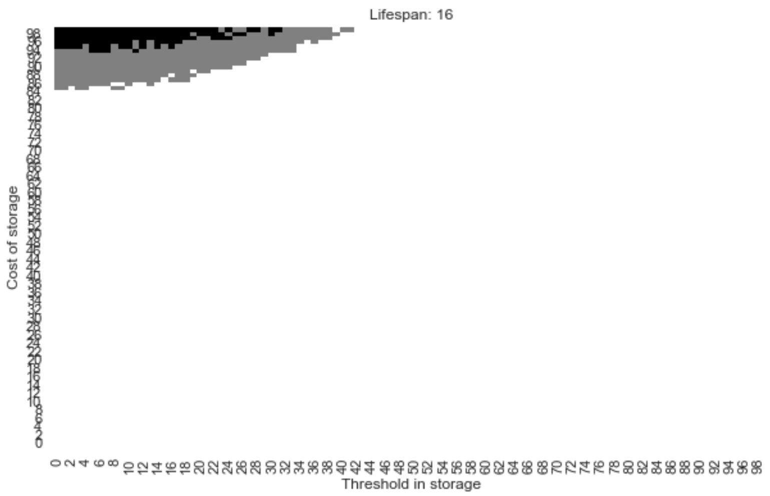


Figure 7.2: The results of an evolutionary game between CR3SR2 and IF3S With a single resource for lifespan 16 years. [Source: Own.]

In Figure 7.2, at 16 interactions, we can see that CR3SR2 barely has any viable parameters. At numbers higher than 23, the benefits of the storage outweigh any of the tested values of costs.

Conclusion

Perceptual researchers typically assume that from an evolutionary point of view it is clearly desirable for an organism to achieve veridical percepts of the world. They assume, that is, that truer perceptions are ipso facto more fit. J. T. Mark et al. [22] tested this assumption using standard tools of evolutionary game theory. They discovered that more realistic perceptions are not necessarily more successful: Natural selection can drive realistic perceptions to extinction when competing with perceptions that use specific interfaces that simplify and adapt the truth in order to better represent the utility of what is being perceived.

Their simulations do not find that natural selection always drives truth to extinction. Therefore, we created conditions in which natural selection gives priority to (simplified) veridical perceptions. We defined strategies that store excessive amounts of the resource and studied an evolutionary game between the strategies IF3 and CR3SR2. Given the reasonably low cost of storage of the resource, innovative simplified veridical perceptions may displace the interface perceptions, even if the latter have the advantage of the first choice of the territory. Furhermore, we examined what happens when veridical strategy with storage and the interface strategy with storage compete. Our simulations show that interface strategy with storage drives the critical realist with storage to extinction.

Using these illustrative examples, we conclude with some open problems. First, we (vaguely) define four stategies A, B, C, D to constitute the valley of death, if A perishes against B, which perishes against C, which perishes against D. In addition, C uses the perceptions of A to support an innovation that cannot be supported using perceptions of B, and D uses the perceptions of B to perceive the true utility of innovation of A.

For the mathematical direction of the research, we conjecture that for each non-monotonous utility function, there exist four strategies that exploit the non-monotonicity of that function to exhibit the technological valley of death.

For microeconomic direction of the research, games with incomplete information could be defined as games where the state of the world is inaccurately perceived by both the agents.

Perception strategies could be introduced into those games so as to either heuristically perceive the expected utility of the situation (modelling heuristics within the current approach to these games), or to add additional information about the state of the world (modelling cheating at such games, or research in market situations). In these cases, such perceptive games could yield improved understanding of the role of marketing, marketing research, and advertising.

For decision science and management direction of the research, the role of perceptions vs. decisions could be further explored in the stated setting. Perceptions play significant role in information systems, linked to data acquisition, data quality, data presentation. Decisions based on that data are significsnt in corporate performance management systems. The models presented here could be used for fundamental research in those settings.

Bibliography

[1] J. J. Armao, *Evolutionary Game Dynamics*, Cooperation and Costly Punishment, Harvard University, 2009.

[2] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fundamenta Mathematicae, 3 (1922): pp. 133-–181.

[3] R. F. Bass, *Real analysis for graduate students*, CreateSpace Independent Publishing Platform, 2014.

[4] D. Bokal, P. Fic: Bazične podlage spremljanja procesnih tveganj, 38. mednarodna konferenca o razvoju organizacijskih znanosti, Portorož, Slovenija, 73–78, 2019.

[5] P. Billingsley, *Probability and Measure*, Third edition, John Wiley & Sons, Inc., United States of America, 1995.

[6] S. Debois, T. Hildebrandt, T. Slaats, M. Marquard, Bridging the valley of death: a success story on danish funding schemes paving a path from technology readiness level 1 to 9, Proceedings of the Second International Workshop on Software Engineering Research and Industrial Practice, Florence (2015), pp. 54–57.

[7] G, Technology Readiness Levels, European Commission (online). (Accessed on 10/04/2019). Accessible at:
https://ec.europa.eu/research/participants/data/ref/h2020/wp/20142015/annexes/h2020wp1415annexgtrlen.pdf

[8] W. S. Gang, *Gauge Invariance Approach to Acoustic Fields*, Springer, Singapore, 2019.

[9] W. Geisler, R. Diehl, A Bayesian approach to the evolution of perceptual and cognitive systems, Cognitive Science, 27(3) (2003) pp. 379—402.

[10] A. Gonzalez, Metric and topological spaces, (Accessed on: 04/10/2019). Accessible at: https://www.math.ksu.edu/ agondem/Ab12-13Metric_files/Metric%20and%20topological%20spaces.pdf

[11] K. E. Gulbrandsen, *Bridging the valley of death: The rhetoric of technology transfer*, Graduate Theses and Dissertations, Iowa State University, Iowa, 2009.

[12] D. Heyer, R. Mausfeld, *Perception and the physical world: Psychological and philosophical issues in perception*, Wiley, New York, 2002.

[13] D. D. Hoffman, The interface theory of perception. Object Categorization: Computer and Human Vision Perspective (2009) pp. 148—165.

[14] D. D. Hoffman, M. Singh, C. Prakash, The interface theory of perception, Psychon Bull Rev 22 (2015) pp. 1480–1506.

[15] D. D. Hoffman, M. Singh, C. Prakash, Probing the interface theory of perception: Reply to commentaries, Psychobn Bull Rev 22 (2015) pp. 1551–1576.

[16] J. Hudson, H. F. Khazragui, Into the valley of death: research to innovation, Drug discovery 18 (2013) pp. 610–613.

[17] M. H. Kalos, P. A. Whitlock, *Monte Carlo Methods, Second, Revised and Enlarged Edition*, Wiley-VCH, Weinheim, 2008.

[18] L. D. Landau, E. M. Lifshitz, *The Classical Theory of Fields, Course of Theoretical Physics 2 (4th ed.)*, Butterworth–Heinemann, 2002.

[19] P. Leitner, Technology Readiness Levels, Impact of Science, and the "Valley of Death" (online). (Accessed on 24/07/2019). Accessible at: http://philippleitner.net/technology-readiness-levels-impact-of-science-and-the-valley-of-death/

[20] T. Lindström, S. Silver, *Avoiding the innovation valley of death; Private sector commercialization of publicly funded research*, Master of science thesis, KTH Industrial Engineering and Management, Stockholm, 2017.

[21] J. C. Mankins, Technology readiness levels, White Paper 6 (1995).

[22] J. T. Mark, B. B. Marion, D. D. Hoffman, Natural selection and veridical perceptions, Journal of Theoretical Biology 266 (2010) pp. 504—515.

[23] S. K. Markham, S. J. Ward, L. Aiman-Smith, A. I. Kingon, The Valley of Death as Context for Role Theory in Product Innovation, The journal of product innovation management 27(3) (2010) pp. 402–417.

[24] E. Nešović, E. B. K. Öztürk, U. Öztürk, k-type null slant helices in Minkowski spacetime, Mathematical Communications, 20 (2015), pp. 83–95.

[25] A. G. Nogales, *Conditional Expectation of a Markov Kernel Given Another with Some Applications in Statistical Inference and Disease Diagnosis*, Dpto. de Matematicas, Universidad de Extremadura, 2018.

[26] M. A. Nowak, K. Sigmund, Evolutionary dynamics of biological games, Science, 303 (2004) pp. 793—799.

[27] S. E. Palmer, *Vision Science: Photons to Phenomenology*, Cambridge, MA: Bradford, 1999. Books/MIT Press.

[28] P. Panangaden, The Category of Markov Kernels, Elsevier, 22 (1999), pp. 171–187.

[29] M. Papadimitrakis, *Notes on Measure Theory*, University of Crete, Crete, 2004.

[30] Z. Pizlo, Y. Li, T. Sawada, R. M. Steinman, *Making a machine that sees like us*, Oxford University Press, New York, 2014.

[31] P. A. Quang, Minkowski spacetime, UNI Hamburg, seminar, n.d.

[32] G. Radnitzky, W. W. Bartley, *Evolutionary Epistemology, Theory of Rationality, and the Sociology of Knowledge*, LaSalle, Illinois: Open Court, 1987.

[33] M. Reeder, Notes on Group Theory, Boston College, Chestnut Hill, MA, 2019.

[34] B. Russell, The problems of philosophy, New York: Oxford, University Press, 1912.

[35] R. Schwartz, *Perception*, MA: Blackwell Publishing, Malden, 2004.

[36] M. Searcoid, *Metric Spaces*, Springer Undergraduate Mathematics Series, Dublin, 2007.

[37] S. Shirali, H. L. Vasudeva, *Metric Spaces*, Springer-Verlag, USA, 2006.

[38] K. Tomaschek, A. Olechowski, S. Eppinger, N. Joglekar, A Survey of Technology Readiness Level Users, INCOSE International Symposium 26 (2016), pp. 2101–2117.

[39] J. W. Weibull, *Evolutionary game theory*, MIT press, Massachusetts, 1997.

[40] D. Wheeler, Symmetry, Notes, Liberty University, 2013 (online). (Accessed on 24/07/2019). Accessible at:
http://www.physics.usu.edu/Wheeler/GenRel2013/Notes/GRKilling.pdf.

[41] D. Wheeler, Noether's theorem, Notes, Liberty University, 2014 (online). (Accessed on 24/07/2019). Accessible at:
http://www.physics.usu.edu/Wheeler/ClassicalMechanics/CMNoetherTheorem.pdf.

List of abbreviations and symbols

$\in$	element of
$\mathbb{N}$	set of natural numbers
$\mathbb{R}$	set of real number
$=$	equal to
$\emptyset$	empty set
$\forall$	for all
$<$	less than
$\leq$	less than or equal to
$\subset$	proper subset
$\otimes$	tensor product
$\sim$	similarity
$\equiv$	identical to
$\{\}$	set
$f(x)$	function of x
(a, b)	open interval
Δ	difference
$\sum$	sum of all values in range of series
$\langle x, y\rangle$	inner product
A^c	complement
$\lim$	limit
$\frac{dy}{dx}$	derivative
$\frac{\partial f(x,y)}{\partial x}$	partial derivative
π	pi, a constant, $\pi \approx 3,14159265358979323846\ldots$
e	Euler's number, $e \approx 2.71828$
CRn	n categories Critical Realist strategy
CRnSRm	n categories Critical Realist with storage
EIT	European institute of Innovation and Technology
IFn	n categories Interface strategy
IFnS	n categories Interface strategy with storage
NASA	National Aeronautics and Space Administration
SRm	critically realistic storage with m categories
SIFm	utilitaristic storage with m categories
A^{-1}	inverse
$P(A)$	probability of event A
$\cap$	intersection
$\cup$	union
$\mathcal{P}(X)$	power set
$\exists$	exists
$>$	greater than
$\geq$	greater than or equal to
$\subseteq$	subset
a^b	exponent
$\times$	multiplication
$:=$	equal by definition
$x!$	factorial
δ	Dirac function
$[a, b]$	closed interval
log	logarithm
$\cdot$	scalar product
$\|\|x\|\|$	norm
$A \setminus B$	relative complement
y'	derivative
$\dot{x}$	time derivative
$\int$	integral
∞	infinity symbol
$argmax$	arguments of the maxima
$nCat$	n categories
ITP	Interface Theory of Perception
EU	European Union
PPP	Private-Public Partnership
TRL	Technology Readiness Level
UK	United Kingdom
USA	United States of America

Razširjen povzetek v slovenskem jeziku

Mnogi filozofi in zazavni raziskovalci raziskujejo odnos med našim zaznavanjem in okoljem. To razmerje med zaznavanjem in objektivno realnostjo imenujemo zaznavna oz. percepcijska strategija. Predstavimo tri ključne teorije po D. D. Hoffmanu [13]. Najbolj preprosta teorija zaznav je naivni realizem, ki verjame, da s čuti zvesto zaznavamo dejanskost, ki ji naše zaznave zato popolnoma ustrezajo. Druga teorija zaznav je kritični realizem, ki oslabi trditev teorije naivnega realizma: zaznavanje zvesto zaznava del realnosti, ne pa vso realnost. Vmesniška teorija (ali teorija namizja) še dodatno oslabi trditev: percepcija na splošno ne zaznava nobenega vidika realnosti (D. D. Hoffman, [13]). V magistrskem delu želim proučiti odnos med temi tremi razredi zaznavnih strategij. Za raziskovanje relativne ustreznosti odnosov med njimi bomo uporabili evolucijske igre. V povzetku se osredotočimo samo na ključne lastne rezultate in njihov kontekst; bazične podlage in z njimi povezane rezultate izpustimo.

Najprej predstavimo evolucijsko igro, ki so jo definirali J. T. Mark et al. [22]. Poglejmo neskončno populacijo agentov, ki naključno vstopajo v pare, ki tekmujejo v igri dveh igralcev za izbrani vir. V tej igri mora vsak agent izbrati eno od t ozemelj. Vsako ozemlje vsebuje r virov (npr. hrano ali vodo), ki zavzema diskretne vrednosti v nizu $V = 1, 2, ..., m$. Naj bo r_T vektor virov na ozemlju T. Koristnost teritorija T je definirana kot

$$u(T) = U(r_T) = \sum_{i=1}^{k} U_i(r_{T,i}). \tag{7.6}$$

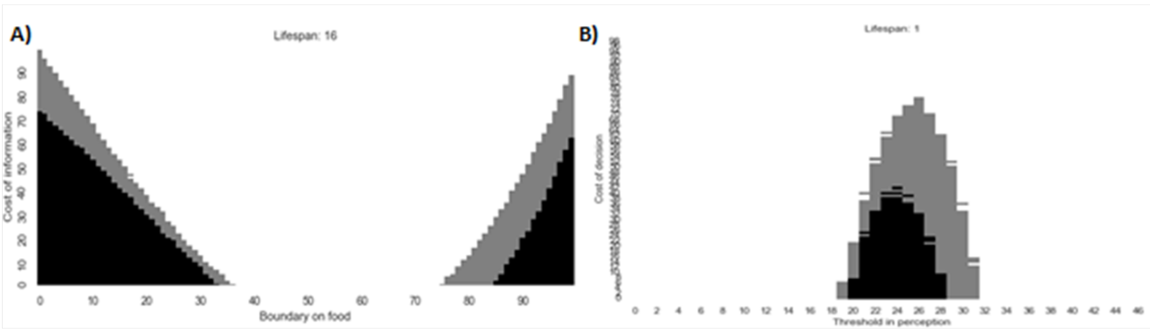
V naših raziskavah je koristnost monotona ali Gaussova funkcija.

Iz J. T. Mark et al.[22] reproduciramo matematični model strategij, ki sodelujejo v tekmovanjih. Ko naivni realist zazna resnične količine virov, $nCat$ kritični realist (CRn) ne zazna resnične količine virov, ampak jo razvrsti v n kategorij, ki ohranjajo urejenost. Mark et al. to imenuje $nCat$ agent. V Tabeli 1 so podane podrobnosti o matematičnih modelih naivnega realista, $nCat$ kritičnega realista in $nCat$ vmesniške strategije.

Elt	**Naivni realist**	**CRn**	**IFn**
W	$\{1,\ldots,m\}^t$	$\{1,\ldots,m\}^t$	$\{1,\ldots,m\}^t$
X	W	$\{0,\ldots,n-1\}^t$	$\{0,\ldots,n-1\}^t$
G	$\{1,\ldots,t\}$	$\{1,\ldots,t\}$	$\{1,\ldots,t\}$
P	w	$\begin{cases} 0; & \text{if } 1 \leq w \leq \beta_1, \\ 1; & \text{if } \beta_1 < w \leq \beta_2, \\ ... & \\ n-1; & \text{if } \beta_{n-1} < w \leq m, \end{cases}$	$\begin{cases} 0; & \text{if } 1 \leq w \leq \beta_1, \\ 1; & \text{if } \beta_1 < w \leq \beta_2, \\ ... & \\ n-1; & \text{if } \beta_{n-1} < w \leq m, \end{cases}$
A	$u' = u + U(r_i), r'_i = 0$	$u' = u + U(r_i), r'_i = 0$	$u' = u + U(r), r'_i = 0$
D	$\text{argmax}_{p=P_n(w), i \in G_n}\, p[i]$	$0 < n-1 < ... < 1$	$argmax_{p=P_i(w), i \in G_i}\, p[i]$
U	$U(r_i)$	$U(r_i)$	$U(r_i)$

Tabela 7.2: Matematični modeli naivnega realista, CRn in IFn.

J. T. Mark et al. [22] so najprej raziskali tekmovanje med naivno realistično in $2Cat$ kritično realistično strategijo (CR2). Za zaznavanje in obdelavo več podatkov je potrebno več energije. Zato veristična strategija porabi več energije za dosego svojega cilja. Poleg tega so raziskovali tekmovanje med $3Cat$ kritičnim realistom (CR3) in $3Cat$ vmesniško strategijo (IF3).

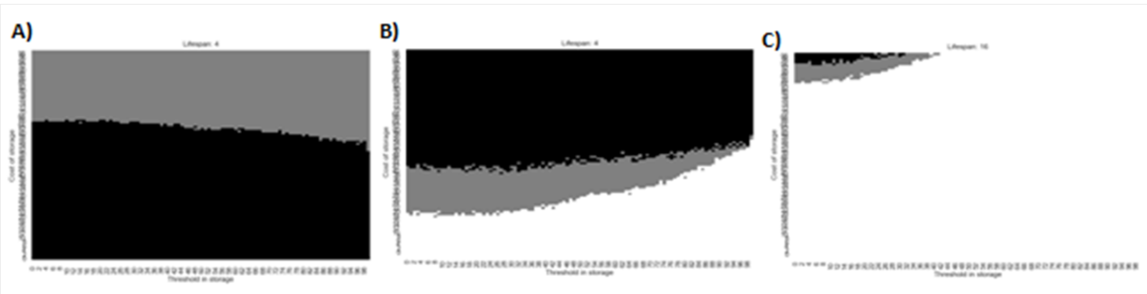


Slika 7.3: A) Rezultati evolucijske igre med naivnim realistom in kritičnim realistom z enim samim virom. B) Rezultati evolucijske igre med CR3 in IF3 z enim samim virom. [Vir: Reproducirano iz [22]]

Slika 7.3 A) prikazuje, kako stroški informacij in prag zaznave, ki ga kritični realist uporablja za dojemanje sveta, vplivajo na tekmovanje kritičnega realista in naivnega realista. Bela barva predstavlja vrednosti parametrov, kjer kritičen realist izpodrine naivnega realista, črna, kjer naiven realist izpodrine kritičnega realista in siva, kjer obe strategiji stabilno sobivata. Kot je razvidno iz slike, kritične realistične strategije lahko izpodrinejo naivno realistično strategijo. Celotna resnica ni vedno uspešnejša od poenostavljene resnice. V nadaljevanju so Mark et al. primerjali naslednje 3 strategije: naivno realistično strategijo, 3Cat kritično realistično strategijo, in 3Cat vmesniško strategijo. Mi reproduciramo njihove rezultate le s tekmovanjem med CR3 in IF3. Na sliki 1.B) lahko vidimo rezultate: vmesniške strategije lahko evolucijsko premagajo kritične realistične strategije. Resnica, v celoti ali poenostavljna, ni vedno uspešnejša od zaznav, ki spoznavanje dejanskosti nadomestijo z neposrednim osredotočanjem na koristnost zaznav.

V nadaljevanju predstavimo lastno nadaljevanje razmislekov in obravnavamo strategije zaznav, ki imajo možnost shranjevanja viška virov. Podlaga za uvedbo takih strategij je evolucijska: neka mutacija organizma IF3, ki sicer utilitaristično zaznava količino vira, lahko razvije možnost shranjevanja prekomerne količine vira. V tem primeru se mora odločiti, ali bi vir potrošila ali shranila in za odločitev potrebuje verodostojno zaznavo o njegovi količini: je evolucijsko motivirana za veristično zaznavo. Količina vira v shrambi se giblje od 1 do 100. Pri izbiri ozemlja je agentov cilj čim večja koristnost uporabljenega vira, višek pa (če je v shrambi prostor), shrani za prihodnost. To pomeni, da če je shramba prazna, agent izbere teritorij z največjo količino vira ter najprej izkoristi del, ki mu prinaša največjo koristnost, potem pa višek shrani. Če pa je shramba polna, potem agent izbere teritorij, ki mu prinaša največjo koristnost. Te strategije igro začnejo s prazno shrambo in jo v igri, kadar je vira v okolju preveč, polnijo. Da bi bilo shranjevanje v celoti izkoriščeno, moramo v model vnesti še en parameter - življenjska doba l spremlja število interakcij med dvema konkurenčnimi strategijama, tj. kako pogosto se lahko dva predstavnika potegujeta za neko ozemlje in izkoriščata prednost hrambe prevelike količine ali dopolnitve nezadostne količine virov.

V Tabeli 2 so podane podrobnosti o matematičnih modelih nCat kritičnega realista s shrambo in nCat vmesniške strategije s shrambo.
V nadaljevanju proučujemo tekmovanja med IF3 in CR3SR2 in med IF3S in CR3SR2.



Slika 7.4: A) Rezultati evolucijske igre med CR3SR2 in IF3 z enim samim virom in življenjsko dobo 4 leta. B) Rezultati evolucijske igre med CR3SR2 in IF3S2 z enim samim virom za življenjsko dobo 4 leta. C) Rezultati evolucijske igre med CR3SR2 in IF3S z enim samim virom za življenjsko dobo 16 let. [Vir: lasten]

Slika 2.A) prikazuje rezultate tekmovanja med IF3 in CR3SR2 za dane vrednosti življenjske dobe $l = 4$ interakcij, pri čemer je korist shranjevanja največja. Vidimo, da CR3SR2

Elt	CR3SR2	IF3S
W	$\{1,\ldots,m\}^{t+1}$	$\{1,\ldots,m\}^{t+1}$
X	$\{0,\ldots,2\}^t \times \{0,1\}$	$\{0,\ldots,2\}^t \times \{0,1\}$
G	$\{1,\ldots,t\}$	$\{1,\ldots,t\}$
P	$P_c(t) = \begin{cases} 0; & \text{if } 1 \leq t \leq \beta_c, \\ 1; & \text{if } \beta_c < t \leq \gamma_c, \\ 2; & \text{if } \gamma_c < t \leq m. \end{cases}$ $P_s(s) = \begin{cases} 0; & \text{if } 1 \leq s \leq \beta_s, \\ 1; & \text{if } \beta_s < s \leq m, \end{cases}$	$P_u(t,s) = \begin{cases} 0; & \text{if } 0 \leq U_c(t,s) \leq \beta_u, \\ 1; & \text{if } \beta_u < U_c(t,s) \leq \gamma_u, \\ 2; & \text{if } \gamma_u < U_c(t,s). \end{cases}$
A	$s' = \begin{cases} \min(s_{max}, r_i + s - r^*); & \text{if } r_i + s > r^*, \\ \max(0, r_i + s - r^*); & \text{if } r_i + s \leq r^*, \end{cases}$ $u' = u + \begin{cases} U(r^*); & \text{if } r_i + s > r^*, \\ U(r_i + s); & \text{if } r_i + s \leq r^*, \end{cases}, r_i' = 0$	$s' = \begin{cases} \min(s_{max}, r_i + s - r^*); & \text{if } r_i + s > r^*, \\ \max(0, r_i + s - r^*); & \text{if } r_i + s \leq r^*, \end{cases}$ $u' = u + \begin{cases} U(r^*); & \text{if } r_i + s > r^*, \\ U(r_i + s); & \text{if } r_i + s \leq r^*, \end{cases}, r_i' = 0$
D	$\begin{cases} 0 < 1 < 2; & \text{if } 1 \leq s \leq \beta_s, \\ 0 < 2 < 1; & \text{if } \beta_s < s \leq m, \end{cases}$	$0 < 1 < 2$
U	$U(r_i)$	$U(r_i)$

Tabela 7.3: Matematični modeli od CR3SR2 in IF3S.

izpodrine IF3 (črna barva) za vse pregledane nizke vrednosti stroškov skladiščenja in soobstaja za vse višje. Čeprav je CR3SR2 pri izbiri ozemlja počasnejši, mu skladiščenje daje prednost pred utilitarističnim IF3, kadar so stroški skladiščenja primerljivi z uporabnostjo terena. Slika 2.B) prikazuje rezultate tekmovanja med CR3SR2 in IF3S. Slika prikazuje uspeh inovativne utilitaristične vmesniške strategije nad inovativnim kritičnim realistom, ki je poudarjena z večjim številom povprečnih interakcij med organizmi: v primerjavi s precej primerljivim uspehom obeh strategij v štirih interakcijah na sliki 2.B), CR3SR2 pri 16 interakcijah na sliki 2.C) komaj ima sposobnost preživeti, pri 24 interakcijah pa ga IF3S izpodrine za vse vrednosti parametrov.

J. T. Mark et al. so v svoji raziskavi odkrili, da bolj realistične percepcije niso nujno uspešnejše. Vendar smo ustvarili pogoje, v katerih naravna selekcija daje prednost (poenostavljenim) verističnim dojemanjem. Definirali smo strategije, ki shranijo prekomerne količine virov, in proučevali evolucijsko igro med strategijama IF3 in CR3SR2. Ob nizkih stroških shranjevanja vira lahko inovativne poenostavljene veristične percepcije izpodrinejo veristične zaznave, tudi če imajo slednji prednost pred prvo izbiro ozemlja. V nadaljevanju smo proučili, kaj se zgodi, ko konkurirata veristična strategija s shrambo in vmesniška strategija s shrambo. Naše simulacije kažejo, da vmesniška strategija s shrambo izpodrine kritičnega realista s shrambo.

Naš model prikazuje tehnološko dolino smrti kot samoporarajoči evolucijski pojav v evolucijskih okoljih, v katerih se agentov model sveta razlikuje od resnične strukture sveta, uporabnost virov pa je po količini virov nemonotona in agenti razvijajo svojo percepcijo (tj.

svoj model), postopek odločanja in svoja dejanja, in pridobivajo evolucijsko prednost pred tistimi, ki bodisi dojemajo manj realistično (torej ne morejo inovirati) bodisi manj utilitaristično (s čimer ne morejo maksimirati evolucijske uporabnosti inovacij).